JOURNAL OF ALGEBRA 71, 527–543 (1981)

Generators, Relations and Coverings of Algebraic Groups, II

ROBERT STEINBERG*

Department of Mathematics, University of California, Los Angeles, California 90024

Communicated by Walter Feit

Received November 12, 1980

1. INTRODUCTION

Our purpose here is to fill the two remaining gaps in the determination of the Schur multipliers of the finite simple Chevalley groups and their twisted analogues by providing 1.1, 5.1 and 6.1 (see also 6.2) below.

THEOREM 1.1. *Let G be the group of rational points of a simply connected simple affine algebraic group defined and split over a finite field of q elements. Then the Schur multiplier M of G is trivial with the following exceptions*: (a) *If G is $A_1(4)$, $A_2(2)$, $A_2(3)$, $C_3(2)$, $F_4(2)$ or $G_2(4)$ then M is $\mathbb{Z}_2$ (cyclic of order 2); if G is $A_1(9)$, $B_3(3)$ or $G_2(3)$ it is $\mathbb{Z}_3$.* (b) *If G is $A_2(4)$* (resp. $D_4(2)$) *then M is the direct product of $\mathbb{Z}_4$ (resp. $\mathbb{Z}_2$) with itself.*

This is proved in Schur [11, p. 119] for $G = SL_2$ and in [13, 4.1] for the oher types with $q \geqslant 5$. Most of the exceptional cases of 1.1 have been treated by other authors (see Sects. 2, 3 below). In Sects. 2, 3 we treat the case $q \leqslant 4$, limit M as indicated, and construct an appropriate covering group when this is easy, using mainly the spin covering which is developed in Section 7. In this regard we call attention to a forthcoming paper of Robert Griess, "Schur multipliers of the known finite simple groups III," in which such a construction is given for each of the simple groups (including the 26 sporadic groups) for which this has not been done here or elsewhere in the literature. The main idea in this part, as in [13], is to attempt to lift a certain presentation of G (see Section 2 below) to an arbitrary central extension.

In the second part of this paper, Sections 4–6, we obtain similar presentations for the finite quasisplit groups $SU_{2n+1}(k/k_0)$. From this, one can determine the Schur multipliers of these groups, and in fact Griess [9] has

* Part of this paper was written in 1979–1980 while the author was a guest at Queen Mary College. He acknowledges with pleasure the extreme hospitality shown him during this visit.

0021-8693/81/080527–17$02.00/0

already done this, among other things. His results together with those of Grover [10] and Alperin and Gorenstein [2] show that if G is twisted in 1.1 then M is trivial except for the following cases: $M({}^2A_3(2)) = \mathbb{Z}_2$, $M({}^2A_3(3)) = \mathbb{Z}_3 \times \mathbb{Z}_3$, $M({}^2A_5(2)) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $M({}^2B_2(8)) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $M({}^2E_6(2)) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Other papers that are relevant to this part are Abe [1] and Deodhar [7], where fragments of our calculations may be found.

Our results were obtained more than 10 years ago (that the number of exceptions in 1.1 is finite is mentioned in [13]) but remained unpublished because of a miseading of the literature on our part. We are indebted to Robert Griess for setting us straight on this point and for helping us otherwise in the preparation of this paper.

2. Start of the Proof of Theorem 1.1

For $G = SL_2$ the result is due to Schur [11, p. 119] and is also proved in [13]. The exceptional cases are covered by the sequences $SL_2(4) \sim PSL_2(5) \leftarrow SL_2(5)$ and $SL_2(9) \to PSL_2(9) \sim A_6$ (alternating group) which has a 6-fold covering group [12]. (Here and elsewhere "$\sim$" denotes an isomorphism.) Hence for the rest of the proof we may assume that the rank is at least 2. The group G is as in 1.1, thus is a universal Chevalley group in the language of [14] (this is a departure from [13], where G denotes the adjoint group), H a split Cartan subgroup, N, $W = N/H$ (Weyl group), $R = \{r, s, t, \dots\}$ the root system, $x_r(u)$ as in [13], as are $X_r = \{x_r(u) \mid u \in k\}$, $w_r(u) = x_r(u)\, x_{-r}(-u^{-1})\, x_r(u)$ $(u \neq 0)$ and $h_r = w_r(u)\, w_r(-1)$. The X_r's taken together generate G and they satisfy the following relations, which in case k is finite as is now being assumed form a complete set [13, 3.3].

$$\text{(A)} \qquad x_r(u)\, x_r(v) = x_r(u+v),$$

$$\text{(B)} \qquad (x_r(u), x_s(v)) = \prod x_{ir+js}(C_{ijrs} u^i v^j).$$

Here u and v run over k and r and $s \neq -r$ over R. The term on the left of (B) is the commutator of the two factors and on the right is the product over all pairs of positive integers (i,j) taken in any order, the C_{ijrs} being certain integers which depend on the order chosen but not on u or v and are known, at least up to sign, in the various cases (see, e.g., [6], [13] or [14]). Now let $\pi\colon G_1 \to G$ be a universal central extension of G (so that $\ker \pi$ is $M(G)$). As stated above our object is to lift relations (A) and (B) to G_1 if this is possible and to measure the obstruction if not. Since this is done for $q \geqslant 5$ in [13] we may assume from now on that $q \leqslant 4$. To compensate for the smallness of k, we shall have to be able to produce roots with suitable geometric properties. The basic result towards this end is as follows.

LEMMA 2.1. *Let R be a root system and S a subsystem closed under real (or rational) linear combinations. Then any basis, i.e., simple system, for S extends to one for R.*

Proof. Let A and B be the real spaces generated by R and by S, and C the orthogonal complement of B in A. Order C arbitrarily, B compatibly with a given basis for S, and then A lexicographically so that $b + c > 0$ if $c > 0$ or if $c = 0$ and $b > 0$. If now s is simple in S and $s = r_1 + r_2$, a sum of positive roots in R, with $r_1 = b_1 + c_1$ and $r_2 = b_2 + c_2$ in $B + C$, then $s = b_1 + b_2$ and $0 = c_1 + c_2$. Thus $c_1 = c_2 = 0$, whence r_1 and r_2 are in S, and s remains simple in R, as required.

We turn now to the proof of 1.1 proper, considering in the rest of this section groups for which there is just one root length.

2.2. We assume first that $q > 2$.

(1) In the present case (R simple, one root length) any two roots are contained in a subsystem of type A_1, A_2 or A_3. To see this, let B be a basis for the system generated over the reals by the given roots and A an extension to R as given by 2.1. If B is not of type A_1 or A_2 then it is of type A_1^2 and we may adjoin to B the root which is the sum of the intermediate roots to those of B in the Dynkin diagram of A (which is connected since R is simple) to get a basis of type A_3. Henceforth we shall omit proofs of this nature.

Now we define $f\colon X_r \to G_1$ as in [13, Sect. 9]: choose $h_r \in H$ so that $r(h_r) \neq 1$, i.e., $x \to (h_r, x)$ is a bijection on X_r, and then define $f(x)$ so that $f((h_r, x)) = (f(h_r), f(x))$ with $f(h_r)$ any lift of h_r. This is not circular since the right side is independent of f.

(2) $\pi^{-1}H$ transforms $f(X_r)$ into itself. More generally, if $n \in N$ corresponds to $w \in W$ then $\pi^{-1}n$ transforms $f(X_r)$ into $f(X_{wr})$. The proof is like that in [13, Sect. 9, step (1)].

(3) Each $f(X_r)$ is Abelian. Write $r = s + t$, the sum of two other roots and $x = (y, z) \in X_r \cap (X_s, X_t)$ accordingly. If $x' \in X_r$, then x' commutes with y and z since $r + s$ and $r + t$ are not roots. Thus $f(x')$ commutes with $f(y)$ and $f(z)$ up to central elements of G_1, thus with $(f(y), f(z))$ exactly, thus also with $f(x)$ which differs from this by a central element of G_1.

(4) The relations (A) hold (for the elements $f(X_r)$). The proof is as in [13, Sect. 9, step (3)].

(5) If r and s are independent roots there exists $h \in H$ such that $r(h) = 1$ and $s(h) \neq 1$ (in k) unless $G = A_2(4)$ or $q = 3$ and r and s are orthogonal. For this assume first that r and s are not orthogonal. If the rank is at least three we choose a root t with $\langle r, t\rangle = 0$ (Cartan integer) and $\langle s, t\rangle = 1$ (this is possible in A_3, hence in general by (1)) and set $h = h_t(c)$ with $c \neq 1$; if $G = A_2(3)$ set $h = h_r(c)$. If r and s are orthogonal and $q = 4$ then $h = h_s(c)$ works.

(6) If $G \neq A_2(4)$ then relations (B) hold. We use induction on the number of roots involved. Let $c(t,u)$ denote the ratio of the two sides of (B) when the x's are replaced by the $f(x)$'s. By (4) and our inductive assumption it easily follows that c is "biadditive." In the favorable cases of (5) we conjugate our relation by $f(h)$, any lift of h. By (5) and the centrality of c we get $c(t, u) = c(t, us(h))$ whence $c(t, u(s(h) - 1)) = 1$ and c is identically 1. If $q = 3$ and r and s are orthogonal and also t and u are both 1, which we may assume since the additive group of k is cyclic, there exists $n \in N$ transforming $x = x_r(1)$ and $y = x_s(1)$ into each other (easily checked in SL_4, see (1).) Thus $(f(x), f(y)) = f(n)(f(x), f(y)) f(n)^{-1} = (f(y), f(x))$ and $c^2 = 1$. But also $c^3 = 1$ since c is additive. Thus $c = 1$, as required.

(7) Now let $G = A_2(4)$. Everything is as before through (4) so that only relations (B) need be considered. In the 60° case ($r + s$ is not a root) the obstruction is biadditive, hence yields an elementary 2-group. In the 120° case if we write (B) (lifted) as $f(x_r(t)) f(x_s(u)) f(x_r(t))^{-1} = c_{rs}(t, u) f(x_{r+s}(tu)) f(x_s(u))$, the same with u replaced by v, and then multiply, we get $(*)$ $c_{rs}(t, u + v) = c_{rs}(t, u)\, c_{rs}(t, v)\, c_{r,r+s}(u, tv)$. This shows (take $v = u$) that $c_{rs}^4 = 1$ and that the 60° obstructions are expressible in terms of the 120° ones. If we make H act and use (2) we get $c_{r,r+s}(t, u) = c_{r,r+s}(tv, uv^{-1})$ and $(**)$ $c_{rs}(t, u) = c_{rs}(tv, uv)$ for all $v \in k^*$, so that in particular $c_{r,r+s}$ is symmetric. Let a be a generator of k^*. Take the product of $(*)$ with $t = a$ and (u, v) running over all pairs of distinct elements of k^*. In view of the above remarks we get $c_{rs}(a, 1)\, c_{rs}(a, a)\, c_{rs}(a, a^2)$ equal to the same item with s replaced by $r + s$, which is 1 by additivity in the last position. Since also the Weyl group is transitive on 120° pairs of roots and $(**)$ holds, the obstruction to lifting all of relations (B), hence to lifting G itself, is reduced to a potential $(4, 4)$ group generated by $c_{rs}(a, 1)$ and $c_{rs}(a, a^2)$. That this obstruction is real has been shown by Burgoyne and by Thompson, unpublished. One way to do this is to construct the central extension over B (the Borel subgroup corresponding to the positive roots) first and then to check certain compatibility conditions for the action of the Weyl group, sufficient for the extension of the construction to the whole group. (See [9, pp. 366–371], where this method is applied to the group $G_2(3)$, mentioned in Sect. 3.5 below.)

2.3. Now we assume that $q = 2$ (and still that there is just one root length). We set $x_r = x_r(1)$ and define $y_r = f(x_r)$ thus. Write $r = s + t$, a sum of two other roots, so that $x_r = (x_s, x_t)$ (one of the relations of (B)) and then $y_r = (y_s, y_t)$.

(1) If the rank is $\geqslant 3$ and $\langle r, s\rangle = 1$ (i.e., r, s make an angle of 60°), then y_r and y_s commute. To see this, write $s = t + u$ with r, t, u independent. Then $r + t$ and $r + u$ are not roots: if, e.g., $r + t$ were a root, we would have $\langle r, t\rangle < 0$, whence $\langle r, u\rangle > 1$, which is impossible since $r \neq u$. Thus y_r

commutes with y_t and y_u up to central elements and thus with y_s exactly (see step (3) of 2.2).

(2) If the rank is $\geqslant 3$, then (A) holds (for the y_r's), i.e., $y_r^2 = 1$. Write $y_r = (y_s, y_t)$ as at the start. Then y_r commutes with y_s and y_t by (1). Thus $y_r^2 = (y_s, y_t^2) = 1$ since y_t^2 is central.

(3) Assume that the rank is $\geqslant 4$ and that the type is A_n or E_n. If r and s are orthogonal roots, then y_r and y_s commute. By 2.2 (1) and our present assumptions we may imbed r and s into a root system of type A_3 and the latter into one of type A_4. There we write $s = t + u$ with all roots orthogonal to r. Then y_r commutes with y_s as in (1).

(4) Assume as in (3). Then the y_r's satisfy conditions (A) and (B). By (1), (2) and (3) it remains only to show that the normalization of y_r is independent of the choice of s and t. Let $r = u + v$ be another such choice. If $u = t$ and $v = s$, the result holds by (2). Otherwise u is linearly independent from s and t, and from $\langle s + t, u\rangle = \langle u + v, v\rangle = 1$, we get one of $\langle s, u\rangle$, $\langle t, u\rangle$, say, the first, equal to 1, the other equal to 0, and vice versa if u is replaced by v. Thus $s - u = v - t$ is a root, call it p, and it is orthogonal to $s + t$. If we conjugate (y_s, y_t) by $w_p \in W$, represented in N by an element of $X_p X_{-p} X_p$, we leave it unchanged by (2) and transform it into (y_u, y_v) since $w_p s = u$ and $w_p t = v$, whence (4).

(5) If the type is A_2 or A_3, then π is a double covering. For A_2 we get in (1) an obstruction $j = (y_r, y_s)$, $j^2 = 1$, independent of r and s (because of the action of W), and then each $x_r^2 = j$ in (2). We can realize the covering concretely via $SL_3(2) \sim PSL_2(7) \leftarrow SL_2(7)$. For type A_3 we have (1) and (2), but in (3) an obstruction $j = (y_r, y_s)$, $j^2 = 1$, independent of r and s, and in (4) an obstruction expressible in terms of j. Here we get our realization via $SL_4(2) \sim A_8$ (alternating group) $\subset SO_8 \leftarrow \mathrm{Spin}_8$ (See 7.9 below.)

(6) The groups $D_n(2)$ $(n \geqslant 4)$ remains to be considered. As above (1) and (2) hold; thus the crux of the matter is (3). In terms of coordinates the roots may be written $\pm v_i \pm v_j$ $(1 \leqslant i < j \leqslant n)$, more simply $\pm i \pm j$. Assume $n > 4$ first. Then for r and s of the form 12 and 34 we write $34 = 35 + (4 - 5)$ to get (3). For the only other possible form 12, $1 - 2$ we set $j = (y_r, y_s)$, which is independent of the indices involved. We conjugate y_{12} by $w_{1-2} w_{34} w_{3-4}$. By our result for pairs of the first form and the fact that w_{1-2} may be chosen in $X_{1-2} X_{2-1} X_{1-2}$ we get $j y_{12}$. If we conjugate in the same way (y_{13}, y_{-32}), equal to y_{12} up to a central element, we get (y_{2-3}, y_{13}), which is (y_{13}, y_{2-3}) by (2). Thus $j = 1$ and we have (3) and hence (4) in this case. If $n = 4$, there are three orbits of orthogonal pairs of roots, represented by $(12, 1-2)$, $(12, 3-4)$ and $(12, 34)$. These yield obstructions j, k and l in (3), each of square 1. We claim that $jkl = 1$. In fact the calculation just made shows that this is so. Further the obstructions in

(4) are expressible in terms of j, k and l. We thus have a potential $(2,2)$ covering which can be realized via $D_4(2) \leftarrow SW(E_8)$ (special Weyl group) $\subset SO_8 \leftarrow \mathrm{Spin}_8$. Here the first arrow is a double covering realized by reduction mod 2.

The proof of 1.1 in the single root case is now complete.

3. The Proof Concluded

3.1. We turn next to the groups $C_n(q)$. We assume first that $n \geqslant 2$ and $q=4$. We normalize $y_r(u) \equiv f(x_r(u))$ as in 2.2 and see as there that

(1) $\pi^{-1}H$ normalizes $\{y_r(u)\}$ for each r.

(2) If $r \neq s$ and $x_r(u)$ and $x_s(v)$ commute, then so do $y_r(u)$ and $y_s(v)$. If r and s are orthogonal we remove any potential obstruction by conjugation by $h_r(c)$ $(c \neq 1)$ as in 2.2 (6). If r and s make an acute angle, there exists a long root t orthogonal to one of them and not to the other: if r is long and s short, then $t=r-2s$ will do, while if both are short we imbed $\{r, -s\}$ in a basis of type C_3 (see 2.1) and use the remaining basis vector. In both cases $h_t(c)$ works as in the first case.

(3) Each $\{y_r(u)\}$ is Abelian. Consider a relation (B) of the form $(y_s, y_t) = c y_{s+t} y_{s+2t}$ with c central. Since y'_{s+t} commutes with y_s, y_t and y_{s+2t} by (2), it also commutes with y_{s+t}; and similarly for y'_{s+2t}. Since every y_r occurs, up to a central factor, in the role of y_{s+t} or y_{s+2t} above, we have (3).

(4) Relations (A) hold. By (1) and (3) as in 2.2.

(5) Relations (B) hold. By (2) we are left with the case in which r and s make an obtuse angle. By (2) again the obstruction c is biadditive, hence may be removed as in the second case of (2).

3.2. Next come the groups $C_n(3)$ $(n \geqslant 2)$. For $n=2$ the central extension $Sp_4(3) \to SU_4(2)$ shows that $\ker \pi$ is both a 2-group and a 3-group (by analogues of 6.2 below which hold for these groups and are proved in the same way (see [13, 4.5]), hence is trivial. Thus we may assume that $n \geqslant 3$. For r short we normalize $y_r(u)$ as in 2.2, while for r long, in which case H is too small for this, we write $r=s+t$ with s and t short so that $x_r(u) = (x_s(u), x_t(\varepsilon))$ for some sign ε and then set $y_r(u) = (y_s(u), y_t(\varepsilon))$.

(1) Ir $r+s$ is not a root then relations (B) hold. If r and s are both short, the relation is within a subgroup of type $A_3(3)$, a case treated in 2.2. If both are long we can write s as a sum of two (short) roots orthogonal to r and proceed as in 2.3 (1), while if r is long and s short we can choose a long root t orthogonal to r and not to s and then use $h_t(-1)$ as in 2.2 (6).

(2) Relations (A) hold. If r is short we go to $A_3(3)$ as before. If r is long, then $(y_s(u), y_t(\varepsilon))$ (as above) commutes with both factors by (1). Thus it is "additive" in the first factor, thus also in u since central factors inside commutators are immaterial.

(3) Relations (B) hold. By (1) we may assume that $r+s$ is a root, say, t. If all roots are short we are inside $A_2(3)$ and so done. If r and s are short and t long, assume first that $t=r+s$ is the decomposition of t used in the normalization of the y_t's, so that (B) holds for the choice u, ε of the parameters. But then it holds for $\pm u$, $\pm\varepsilon$ since the commutator on the left commutes with both of its factors by (1), i.e., for all values of the parameters since $q=3$. If $t=r'+s'$ is another decomposition of t of the present type, there exists a root p orthogonal to t sch that $w_p r=r'$ and $w_p s=s'$. Transforming $y_t=(y_r, y_s)$ by w_p represented in the form $y_p y_{-p} y'_p$ in G_1, we get y_t on the left by (1) and some $(y_{r'}, y_{s'})$ on the right, so that (B) holds in this case. Finally, if r is long and s short, we get in (B) an obstruction which is biadditive by what has already been proved and so may be removed as in the last case of (1).

3.3. For type C_n only the groups $C_n(2)$ $(n \geqslant 3)$ remain to be treated. If r is short we define $y_r=y_r(1)=(y_s, y_t)$ with $r=s+t$ a sum of short roots. If r is long we write $r=s+2t$ and then define $y_r=y_{s+2t}$ by $(y_s, y_t)=y_{s+t}y_{s+2t}$. We assume first that $n \geqslant 5$.

(1) The relations (A) and (B) involving only short roots hold. This is because the short roots support a subgroup of G isomorphic to $D_n(2)$, a case treated in 2.3.

(2) y_r and y_s commute if r is long and $r+s$ is not a root. In terms of the usual coordinates for roots for C_n, the possibilities, up to the action of the Weyl group, are: $r=2v_1$ and $s=v_1-v_2, v_2-v_3$ or $2v_2$. In the first case we write $v_1-v_2=(v_1-v_3)+(v_3-v_2)$ and proceed as in 2.3 (1). In the second case there is a potential obstruction $j=(y_r, y_s)$, $j^2=1$, invariant under W. We apply $w_r w_s$ to $(y_a, y_b)=cy_{a+b}$ with $a=v_2-v_1$, $b=v_1+v_3$ and c central. The left side remains unchanged by (1) since the two terms there get interchanged up to central factors, while the right side gets multiplied by $j^3=j$ since w_s has no effect by (1) and w_r may be chosen as $y_r y_{-r} y'_r$. Thus $j=1$. In the third case we set $s=2v_3+2(v_2-v_3)=a+2b$, say. Then in $(y_a, y_b)=cy_{a+b}\,y_{a+2b}$ the left side commutes with y_r as does the second term on the right by the second case just treated; thus so does the final term, as required.

(3) If r is long relations (A) hold. We square the equation used to define y_r. By (1) and (2) we get $(y_s^2, y_t)=1$ on the left and $y_{s+2t}^2=y_r^2$ on the right.

(4) Relations (B) hold. Because of (1) and (2) this amounts to

showing that for r long our normalization of y_r is indepenent of s and t. If s' and t' form another possibility then up to the action of W there is just one case:

$$r = 2v_1 = 2v_2 + 2(v_1 - v_2) = 2v_3 + 2(v_1 - v_3).$$

If we apply the reflection corresponding to $v_1 - v_2$ to the first normalization we get the second one with y_r intact by (1) and (2), whence our result.

(5) We now assume that n is 3 or 4. We have arranged the steps above so that they all apply modulo the first, i.e., so that all obstructions are expressible in terms of those found for $D_n(2)$ in 2.3. If $n = 4$ the obstructions can be removed. For since the orbits through (12, 3–4) and (12, 34) near the end of 2.3 fuse, $k = l$, whence $j = 1$; and conjugating $(y_r, y_s) = cy_{r+s}y_{r+2s}$ with $r = 2v_4$ and $s = v_3 - v_4$ by y_t with $t = v_1 + v_2$ we get $l = 1$. This requires knowing the second case of (2) which can be proved directly for all $n \geqslant 4$ by writing $v_2 - v_3 = (v_2 - v_4) + (v_4 - v_3)$ and proceeding as in 2.3 (1). If $n = 3$, the apparent obstruction of order 2 is actual as is shown by the sequence $C_3(2) \sim SW(E_7) \subset SO_7 \leftarrow \mathrm{Spin}_7$.

(6) The case $n = 2$ is not covered by our theorem, but we observe that $C_2(2) \sim S_6$ (symmetric group), so that there is a double covering, as was shown by Schur himself in [12].

3.4. Because of the isomorphisms $B_n \sim C_n$ if q is even, $B_2 \sim C_2$ always, the groups $B_n(3)$ $(n \geqslant 3)$ are the only classical ones still to be considered. Here we can normalize all y_r's as in 2.2.

(1) The relations (A) and (B) that involve only long roots hold. For the long roots support a subgroup $D_n(3)$ for which the result is already known.

(2) If r is short the relations (A) hold. Since k is cyclic $\{y_r(u)\}$ is Abelian. Thus (2) follows as in 2.2 (4).

(3) Relations (B) hold if we exclude the single case: r long, s short and orthogonal to r, $n = 3$. By induction on the number of roots involved the obstruction is biadditive and hence may be removed as in 2.2 (6) by conjugation by $h_t(-1)$ with t a (long) root orthogonal to r but not to s.

(4) If $n = 3$ tere is a potential obstruction of order 3 in the excluded case of (3), independent of (r, s) because of the action of W. The obstruction here is real (Fischer, unpublished): $B_3(3)$ can be imbedded in ${}^2E_6(2)$ (adjoint group) (this is the hard part) and the index is prime to 3. Thus we get a nontrival triple covering of $B_3(3)$ by restricting that of ${}^2E_6(2)$ (adjoint group) given by the universal group of the same type.

3.5. We conclude with the two remaining types, first F_4. If $q \geqslant 3$, we normalize $y_r(u)$ as in 2.2; then each of the relations (A) and (B) occurs in a subgroup of type B_4 or C_3, hence holds by what has been done in 3.1, 3.2 and 3.4 (and an extra bit of argument because the normalization here is different from that in 3.2). If $q=2$, we can respect the outer automorphism interchanging long roots and short roots by setting $y_r = y_r(1) = (y_s, y_t)$ with all roots of the same length, whether r is long or short. Using our results for $D_4(2)$ as imbedded in $C_4(2)$ we get all relations in which only roots of one length are involved and also those in which the right side of (B) is 1. We are thus left with a potential obstruction j of order 2 coming from the relations of the form $(y_r, y_s) = jy_{r+s}y_{r+2s}$, a real obstruction as has been shown by Griess [9 pp. 374–379].

For the final type of group G_2 the situation is tighter and the relations of type (B) are more complicated. Since the details are available in [9, pp. 357–371], we omit them here. The results are: for $q=3$ a triple cover, for $q=4$ a double cover. Griess' existence proof in the latter case is very clever and is contained in Section 7. below.

The proof of 1.1 is hereby completed.

4. Split BN Pairs of Rank 1

In this section G is a group of rank 1 with split BN pair, $B = HX$, and $B^- = HY$ opposite to B satisfying:

(1) X and Y (together) generate G.

(2) $N' \equiv (N - B) \cap XYX$ generates N.

(3) $Y \cap B = 1$.

These properties, which are not independent, are enough to ensure uniqueness in the Bruhat normal form. They hold for SL_2, the quasisplit SU_3 (property (2) is verified just after 5.3 below) and the other rank 1 groups arising in the theory of algebraic groups, including the Suzuki and Ree groups. We consider the following relations on the elements of X and Y which hold in G.

(A) Those that hold in X; those in Y.

(B′) Those of the form $w = xyx'$ as in (2) above, thought of as a definition of w in terms of x, y, x'; those of the form $w_{x''=y''}$.

(C) Those of the form $w = w'$ in N'.

(D) The further relations on the elements of N' needed for an abstract definition of N.

Remark 4.1. If $w \in N'$ then $w^{-1} \in N'$ by (B′), whence N' transforms Y into X and $w = {}^w w \in YXY$. Thus relations (B′) are entirely symmetric in X and Y.

PROPOSITION 4.2. (a) *Relations* (A) *to* (D) (*on the set* $X \cup Y$) *define* G *as an abstract group.* (b) *Relations* (A) *and* (B′) *define a group* G_1 *which is a central extension of* G.

Proof. Let H_1 be the subgroup of G_1 consisting of even products of elements of N'. Then H_1 transforms X and Y into themselves, in fact by the same equations as in G, and every $y \in Y - 1$ has the form xwx', both by (B′) and 4.1. It readily follows that if w_0 is a fixed element of N' then $XH_1 \cup XH_1 w_0 X$ is invariant under right multiplication by X and Y and hence equals G_1, and that the kernel of the natural map $\pi: G_1 \to G$ is contained in H_1. Now each $h \in \ker \pi$ transforms X and Y in G_1 as $\pi h = 1$ does in G, i.e., trivially, hence lies in the center of G_1, whence (b). The extra relations (C) and (D) make π injective on H_1, hence also on G_1 since $\ker \pi \subset H_1$, whence (a).

LEMMA 4.3. *In* G_1 (*or in* G) *let* (∗) $w = xyx'$ *as in* (B′). (a) *Then also* $w = x'(w^{-1}xw)(w^{-1}yw)$ *and* $w = {}^w y {}^w x' x$ *in* XYX. (b) *Each of* x, y, x' *is different from* 1. (c) *Conversely for each* x, y *or* x' *different from* 1 *the other items in the equation* (∗) *exist and are unique.* (d) *If we write* $w = w(x)$ *in* (c) *then* $w(x) = w(x')$. (e) *Further* $w(x^{-1}) = w(x)^{-1}$.

Proof. (a) It is easily checked that each equation here is formally equivalent to (∗). (b) $y \neq 1$ since $w \notin X$ and similarly for x and x' by the equations in (a). (c) Given any $y \neq 1$ then $y \notin B$ and hence $y = xwx'$ (Bruhat form) for unique x, x' and w, whence $w = x^{-1}y(x')^{-1}$. Given $x \neq 1$ write $x = ywy'$, whence $w = x(y')^{-1}(w^{-1}y^{-1}w)$ with y, w, y' uniquely determined by x; similarly for x'. (d) By each equation in (a). (e) Take inverses in (∗) and use (d).

LEMMA 4.4. *In* G_1 *assume that* x_1, x_2, x_3 *are in* X *and different from* 1 *and that* $x_2 = x_3 x_1$. *Write* $w(x_i) = x_i y_i x_i'$ *for* $i = 2, 3$ *as in* 4.3(∗) *and* $w(x_1) = x_1'' y_1 x_1$ *as in the second equation of* 4.3 (a). *Then* $y_1 y_3 \neq 1$, *and if* x_4 *corresponds to* $y_4 \equiv y_1 y_3$ *as* x *does to* y *in* 4.3 (c) *then* $w(x_1)\, w(x_2)^{-1}\, w(x_3) = w(x_4)$.

Proof. Writing w_i for $w(x_i)$ and $x_5, x_6, \ldots$ for elements of X whose precise values are not needed we have:

$$\begin{aligned}
w_1 w_2^{-1} w_3 &= w_1 y_2^{-1} x_2^{-1} (w_2 (x_2')^{-1} w_2^{-1})\, w_3 && \text{(def. of } w_2\text{)}\\
&= w_5 w_1 x_2^{-1} w_3 x_6 && \text{(with } x_5 = w_1 y_2^{-1} w_1^{-1}\text{)}\\
&= x_5 x_1'' y_1 x_1 x_2^{-1} x_3 y_3 x_3' x_6 && \text{(def. of } w_1, w_3\text{)}\\
&= x_7 y_1 y_3 x_8 && \text{(since } x_1 x_2^{-1} x_3 = 1\text{)}\\
&= x_7 y_4 x_8 && \text{(def. of } y_4\text{).}
\end{aligned}$$

It follows that $x_7 y_4 x_8 \in N'$ in G, hence also in G_1. Then $y_4 \neq 1$ by 4.3 (b) and $x_7 = x_4$ by the definition of x_4, so that $x_7 y_4 x_8 = w(x_4) = w_4$, as required.

5. The Quasisplit SU_3

In this section G is as just described relative to a separable quadratic extension o fields k/k_0 and a split Hermitian form which may be taken as $u_1\bar{u}_3 - u_2\bar{u}_2 + u_3\bar{u}_1$ in terms of suitable coordinates. As usual X (resp. Y, N, H) is the subgroup of superdiagonal unipotent (resp. subdiagonal unipotent, monomial, diagonal) elements of G. Our goal is the following result.

Theorem 5.1. *Assume that $G = SU_3$ in* 4.2 (a) *and that k is finite. Then relations* (C) *and* (D) *may be omitted. In other words,* (A) *and* (B′) *suffice to define G abstractly.*

The proof requires several steps. At the start k need not be finite.

For x to be in X it must be of the form $(1, a, b;\ 0, 1, \bar{a};\ 0, 0, 1)$ (with the rows of x written out in order) with

5.2. $a\bar{a} = b + \bar{b}$.

We write $x(a, b)$ for this element. Then $x(a, b)\,x(c, d) = x(a + c, b + a\bar{c} + d)$ and $x(a, b)^{-1} = x(-a, \bar{b})$. Further $x(a, b) \neq 1$ just when $(a, b) \neq (0, 0)$, i.e., when $b \neq 0$ by 5.2; and similarly for Y. For $x = x(a, b)$ equation 4.3 $(*)$ works out to

5.3. $w(a, b) = x(a, b)\, y(-\bar{a}\bar{b}^{-1}, \bar{b}^{-1})\, x(ab^{-1}\bar{b}, b)$.

This therefore is our definition of $w(a, b)$ in the abstract group G_1 (in which all of the calculations of this section are taking place). Now $w(a, b)$ in G is just the offdiagonal matrix $[b, -b^{-1}\bar{b}, \bar{b}^{-1}]$. Thus in view of 5.2 condition (2) at the start of Section 4 amounts to: every $b \in k^*$ is a product of such elements having traces that are norms. If k is finite this is clear since then every element of k_0 is a norm. If k is infinite two elements suffice: bj and j^{-1} with $j = b - \bar{b}$ or any nonzero skew element according as $b - \bar{b}$ is nonzero or not.

By 5.3 equation 4.3 (d) becomes

5.4. $w(a, b) = w(ab^{-1}\bar{b}, b)$.

And Lemma 4.4 now reads:

Lemma 5.5. *In G_1 we have that $w(a_1, b_1)\, w(a_2, b_2)^{-1}\, w(a_3, b_3)$ is of the form $w(a_4, b_4)$ if b_1, b_2, b_3 are nonzero, $a_2 = a_1 + a_3$ and $b_2 = b_1 + b_3 + \bar{a}_1 a_3$. Then $a_4 = b_1 b_2^{-1} b_3 (a_1 \bar{b}_1^{-1} + a_3 b_3^{-1})$ and $b_4 = b_1 b_2^{-1} b_3$.*

Proof. The first set of equations follows from $x_2 = x_3 x_1$, the second from $y_4 = y_1 y_3$. (From 5.3 it follows that $y_3 = y(-\bar{a}_3 \bar{b}_3^{-1}, \bar{b}_3^{-1})$ and from 5.3 and a simple calculation that $y_1 = y(-\bar{a}_1 b_1^{-1}, \bar{b}_1^{-1})$, whence $y_4 = y_1 y_3 = y(-\bar{a}_1 b_1^{-1} - \bar{a}_3 \bar{b}_3^{-1}, \bar{b}_1^{-1} + \bar{b}_3^{-1} + a_1 \bar{b}_1^{-1} \bar{a}_3 \bar{b}_3^{-1})$. Then $x_4 = x(a_4, b_4)$ with $\bar{a}_4 \bar{b}_4^{-1} = \bar{a}_1 b_1^{-1} + \bar{a}_3 \bar{b}_3^{-1}$ and $\bar{b}_4^{-1} = \bar{b}_1^{-1} + \bar{b}_3^{-1} + a_1 \bar{b}_1^{-1} \bar{a}_3 \bar{b}_3^{-1}$ by 5.3, whence the expressions for a_4 and b_4 after some simplication using the equation for b_2 already established.)

We now specialize (a_2, b_2) to $(0, j)$ with j a fixed nonzero skew element of k and set $c = j^{-1} b$ and $h(a, c) = w(a, jc)\, w(0, j)^{-1}$. Observe that $h(0, 1) = 1$. Conditions 5.2 and 5.4 become:

5.6. $a\bar{a} = j(c - \bar{c})$.

5.7. $h(a, c) = h(-ac^{-1}\bar{c}, c)$.

If in 5.5 we solve for a_3, b_3, a_4, b_4 in terms of a_1, b_1 and drop the subscript 1 then that relation becomes

5.8. $h(a, c)\, h(-a, 1 - \bar{c}) = h(-ac\bar{c}^{-1}, c(1 - \bar{c}))$.

Now $h(a, c)$ in G works out to $\operatorname{diag}(c, c^{-1}\bar{c}, \bar{c}^{-1})$. Thus in view of 5.6 relations (C) become

(C′) $h(a, c)$ depends only on c, i.e., does not change when a is multiplied by an element of norm 1.

We now reinstate our assumption that k is finite in 5.1. Then each element of k_0 is a norm so that each $c \in k^*$ is allowable in 5.6. Thus the relations (D) require, in addition to (C′), that:

(D′) $h(\cdot, c)\, h(\cdot, d) = h(\cdot, cd)$ for all $c, d \in k^*$ and some choice of the dots.

It remains to show that (C′) and (D′) hold in G_1.

5.9. In a relation (D′) that holds in G_1 all of the dots may be multiplied by any $u \in k_0$ of norm 1.

Let σ be the automorphism of G given by conjugation by $\operatorname{diag}(u, 1, u)$. It acts on X, Y, N',..., hence also on relations (A) and (B′) and so yields an automorphism of G_1. Applying σ to (D′) we get the same relation with each dot multiplied by u.

5.10. We have $h(a, c)\, h(0, d\bar{d}) = h(ad^2\bar{d}^{-1}, cd\bar{d})$.

We conjugate $h(a, c) = w(a, jc)\, w(0, j)^{-1}$ by $h = h(\cdot, d)$. The left side remains unchanged since H_1 is a central extension of the cyclic group H (isomorphic to k^*) and hence is Abelian; on the right side we use ${}^h w(a, b) = w(ad^2\bar{d}^{-1}, bd\bar{d})$ which follows from 5.3 and the fact that h acts on X and Y

by the same formulas in G_1 as in G. The result is an equation which works out to 5.10.

5.11. (C′) holds if c is a generator of k^*.

We set $d = c^{-1}\bar{c}$; it generates te group of elements of norm 1 in k^*. Because of 5.10 we may multiply a by d^3 without changing $h(a, c)$, and because of 5.7 by $-d$, hence also by $d^3(-d)^{-2} = d$, hence by any element of norm 1.

5.12. Relations (D′) hold.

If c or d is a norm, i.e., is in k_0^*, then (D′) holds by 5.10. Assume next that none of c, d, cd is in k_0^*. Then the equation $pc + q\bar{d} = 1$ has a solution with $p, q \in k_0^*$. Set $p = r\bar{r}$ and $q = s\bar{s}$. Then $h(0, p)\, h(a, c)\, h(b, d)\, h(0, q) = h(ar^2\bar{r}^{-1}, cr\bar{r})\, h(bs^2\bar{s}^{-1}, ds\bar{s})$ by 5.10; write this as $h(A, C)\, h(B, D)$. We have $D = 1 - \bar{C}$ by the choice of p and q, and $A\bar{A} = j(C - \bar{C}) = j(D - \bar{D}) = B\bar{B}$ by 5.6, so that $-AB^{-1}$ has norm 1. Replacing b by $-AB^{-1}b$ at the start we achieve $B = -A$ at the end. Thus the original product equals $h(\,\cdot\,, CD)$ by 5.8, then $h(\cdot, pcqd)$ by the definition of C and D, and then $h(0, p)\, h(\cdot, cd)\, h(0, q)$ by 5.10, whence 5.12 in this case. Finally if $c, d \notin k_0$ and $cd \in k_0$ choose a generator e of k^* so that de $\notin k_0$: if e is any generator then either e or e^{-1} will work since otherwise $e^2 \in k_0^*$ and k^*/k_0^* has order at most 2, which is impossible. We also have e, $cde \notin k_0$. Thus by cases already done and 5.9 we have, for suitable choices of the dots, $h(\cdot, c)\, h(\cdot, d)\, h(\cdot, e) = h(\cdot, c)\, h(\cdot, de) = h(\cdot, cde) = h(\,\cdot\,, cd)\, h(\cdot, e)$. The two $h(\cdot, e)$'s here are equal by 5.11, whence 5.12 in this last case.

5.13. Relations (C′) hold.

Given $h(a, c)$, write $c = c_0^n$ with c_0 a fixed generator of k^*. Then $h(a, c)$ is a product of n elements of the form $h(\cdot, c_0)$ by 5.12 and 5.9, hence depends only on n by 5.11, hence only on $c = c_0^n$, as required.

Since relations (C′) and (D′), and hence also (C) and (D), have been shown to hold in G_1, the proof of 5.1 is now complete.

Remarks 5.14. Theorem 5.1 also holds for SL_2 (see [13, 3.3] whose proof provides a model for the present proof) and probably also holds for the Suzuki groups and the Ree groups. It does hold if $q \neq 2, 8$ in the first case and $q \neq 3$ in the second. For then it can be proved that G and G_1 are perfect (easy) and that G has trivial Schur multiplier (one prime at a time, as Schur did for $SL_2(q \neq 4, 9)$, see [2],), whence the central extension of 4.2 (b) is trivial. The same method works for the current groups SU_3 if $q \neq 4$. Unfortunately, most of the omitted cases above are needed in the treatment of groups of higher rank, as in the next section.

6. The Quasisplit SU_{2n+1} $(n > 1)$

In this section G is of this type. It is generated by unipotent subgroups X_r each isomorphic to k (additive group) or to the subgroup X of SU_3 of Section 5. Each X_r has an "opposite" X_{-r} which with it generate a group isomorphic to SL_2 or to SU_3 in these two cases. In G the following relations on the X_r's hold.

(A) For each r the relations in the group X_r.

(B) For $r \neq -s$ those of the form $(x, y) = \prod Z_{ij}$ analogous to (B) in Section 2.

(C), (D) As in Section 4 with (X, Y) replaced by (X_r, X_{-r}) for each r.

For the details of (B) the reader may consult [9, p. 388]. Our goal is the following result.

THEOREM 6.1. *Let G be as above, the quasisplit SU_{2n+1} with $n > 1$.* (a) *Relations* (A) *to* (D) *define G as an abstract group.* (b) *Relations* (A) *and* (*B*) *define a group G_1 which is a central extension of G.* (c) *If the base field is finite, i.e., if q is so, then* (A) *and* (B) *suffice in* (a), *i.e., they imply* (C) *and* (D).

Proof. Since the proof is close to that for split groups given in [13, 3.3] we shall be sketchy. Relations (A) and (B) restricted to positive (negative) roots define abstractly the maximal unipotent subgroups $X(Y)$ of G (see [13, 7.1]). They also imply the relations (B′) of Section 4 for each r and in fact that the w's in all of the N_r''s transform all of the X_s's in G_1 exactly as in G (see [13, 7.2] for the argument). If N_1 denotes the subgroup of G_1 generated by all of the N_r''s then $G_1 = XN_1X$ readily follows and from this (b), as in the proof of 4.2 (b) above. The extra relations (C) and (D) then permit the identification of N_1 with N, the corresponding subgroup of G, whence (a). Finally, if k is finite then for each rank 1 subgroup $\langle X_r, X_{-r}\rangle$ of G_1 we have the relations (A) and (B′) of Section 4, as already noted. By 5.1 and the corresponding result for SL_2 (see [13, 3.3]) we also have (C) and (D), whence (c).

COROLLARY 6.2. *If k is finite in* 5.1 *or in* 6.1 *and p is its characteristic then the p'-part of the Schur multiplier of G is trivial.*

Proof. Here also we essentially follow [13]. We use the easy fact that (∗) a central extension of a finite p-group by a p'-group always spits (see, e.g., [15, Theorem 2.5] where a more general result of Schur is proved). Now let $\pi: G_1 \to G$ be a central extension with $\ker \pi$ a p'-group. In any one of the relations in (A) or in (B) the elements involved all lie in a p-subgroup of G.

Thus by $(*)$ that relation can be lifted from G to G_1. In the process each element of G involved is lifted to the unique p-element of G_1 above it. Thus all of the relations can be lifted together. By 5.1 or 6.1 this yields a splitting map for π, whence 6.2.

7. The Spin Covering

Throughout this section let V be a vector space of finite dimension $n \geqslant 2$ over $\mathbb{R}$ and $(\ , \ )$ a positive-definite inner product on it. Our object is to give a quick self-contained introduction to the spin group in this situation, enough to prove 7.7 below. Other, more comprehensive, treatments may be found in [3–5]. The Clifford algebra $C = C(V)$ is, by definition, the associative algebra (with 1) generated by the linear space V and the relations $(*)$ $v^2 = (v, v)$ for all $v \in V$. In terms of an orthonormal basis $\{v_i\}$ of V these relations become $(**)$ $v_i^2 = 1$ and $v_i v_j = -v_j v_i$ if $i \neq j$. If for each subset S of $\{v_i\}$ we take the product v_S of its elements with the subscripts in increasing order, we get a basis for C which thus has dimension 2^n. For, first by use of the relations every element of C can be reduced to a linear combination of v_S's. To prove linear independence we may do so over $\mathbb{Z}$ since the structural coefficients are all in $\mathbb{Z}$, hence over $\mathbb{F}_2$ on reduction mod 2. By $(**)$, C is now the direct product of n subalgebras, the ith generated by v_i subject to the condition $v_i^2 = 1$, thus having 1 and v_i linearly independent. In the full algebra the 2^n products v_S are thus linearly independent, as required.

In C we have the subalgebra C^+ of even elements, generated by the products $v_i v_j$; it has dimension 2^{n-1}. For example, if $n = 2$, C_+ is the field of complex numbers, and if $n = 3$ it is the skew field of quaternions.

7.1. Center(C) $\cap\, C^+ = \mathbb{R}$.

If $x = \sum c_S v_S \in$ Center (C), we have $v_i x v_i^{-1} = \sum c_S v_i v_S v_i^{-1}$ with $v_i v_S v_i^{-1} = v_S$ or $-v_S$ according as $S - \{i\}$ is of even or odd size. For S even and nonempty the minus sign holds for $i \in S$, whence $c_S = 0$.

On C there is a unique antiautomorphism $x \to x^*$ fixing V, for there is certainly one on the tensor algebra $T(V)$ and it preserves the relations $(*)$ that define C. The product x^* is generally not in $\mathbb{R}$ but it is so if x is decomposable ($x = u_1 u_2 \cdots u_k$ with each u_i in V) and then defines a norm which extends the given norm on V (imbedded in C in the obvious way) and is multiplicative, as is easily checked. We define Spin(V) (or Spin_n) to be the group of such x for which k is even and $xx^* = 1$, or, equivalently, each u_i is a unit vector.

7.2. If u is a unit vector in V, it preserves V by conjugation and acts there as minus the reflection corresponding to u.

For $uuu^{-1} = u$, while if v is orthogonal to u then $uvu^{-1} = -v$ by (**). We denote this action and its extension to Spin(V) by π.

7.3. Every element x of $O(V)$ is a product of at most n reflections.

If 1 is n eigenvalue of x then by induction x is a product of at most $n-1$ reflections. If 1 is not one then from $\det(x-1) = \det(-x)\det(x^t - 1) = \det(-x)\det(x-1)$, we get $\det(x) = (-1)^n$ so that replacing x by xr with r any reflection puts us back in the first case.

THEOREM 7.4. *If* $\pi\colon \mathrm{Spin}(V) \to GL(V)$ *is as above then* $\operatorname{im}\pi$ *is* $SO(V)$ *and* $\ker\pi$ *is the central subgroup* $\{\pm 1\}$.

The first point is by 7.2 and 7.3. If $x \in \ker\pi$ then x commutes with every $v \in V$, hence lies in Center(C) $\cap\, C^+$, hence is a scalar by 7.1, ± 1 since its norm is 1; and both cases occur: $-1 = v(-v) \in \mathrm{Spin}(V)$ for any unit vector v.

Remarks 7.5. (a) By 7.3 and 7.4 every element of Spin(V) is a product of at most n elements of V. Since the unit vectors of V form a compact connected set it follows that Spin(V) is a compact connected Lie subgroup of C^*. That $\pi\colon \mathrm{Spin}(V) \to SO(V)$ is the universal covering requires further argument (see [4]). (b) It also follows that Spin(V) could have been defined as the commutator subgroup of the group of those elements of $(C^+)^*$ that conjugate V into itself.

COROLLARY 7.6. *An involution x of $SO(V)$ with -1 an eigenvalue of multiplicity $2k$ lifts to an element of order 4 in* Spin(V) *if k is odd, to one of order 2 if k is even.*

If $v_1, v_2, \dots, v_{2k}$ are the first elements of an orthonormal basis of V then $\pi(v_1, v_2, \dots, v_{2k})$ may be taken as x. Since $(\pm v_1 v_2 \cdots v_{2k})^2 = (-1)^k$ by (**) we have 7.6.

COROLLARY 7.7. *If G is a subgroup of $SO(V)$ containing an involution as in 7.6 with k odd then $\pi\colon \pi^{-1}G \to G$ does not split. In particular this so for* $\pi\colon \mathrm{Spin}(V) \to SO(V)$.

This follows at once from 7.6.

COROLLARY 7.8. *If n is odd* Spin(V) *has center of order 2. If n is even the center has order 4 and it is cyclic just when $n/2$ is odd.*

This follows from 7.6 applied to the center of G which is 1 if n is odd, $\{\pm 1\}$ if n is even.

COROLLARY 7.9. *If G s a subgroup of A_n (alternating group) containing* *an involution which is the product of 2k disjoint transpositions with k odd then G has a nonsplit 2-fold central extension.*

We apply 7.7 to the obvious imbedding $A_n \subset SO_n$.

This result goes back to Schur [12]. He proved it by in effect explicitly constructing the fragment of the spin group lying above A_n (and of the pin group lying above S_n).

EXAMPLES 7.10. (a) Corollary 7.9 was used for $n = 8$ in 2.3 and could have been used for $n = 5$ in Section 2 since $SL_2(4) \sim A_5$. (b) Corollary 7.7 applied to the groups $SW(E_n)$ $(n = 7, 8)$ is used in 3.1 and 2.3. The case $n = 6$ figures indirectly in 3.2 since $SW(E_6) \sim SU_4(2) \sim PSp_4(3)$. (c) Here is the pretty way in which Griess [9, pp. 363–364] gets a nonsplit double covering of the group $G = G_2(4)$ of 3.5. Let b be the simple short root and P the corresponding parabolic subgroup. Then G acts on G/P and it turns out that $x_a(1)$ has 25 fixed points, thus acts as a product of $(1365 - 25)/2 = 670$ disjoint transpositions, so that 7.9 applies.

REFERENCES

1. E, ABE, Coverings of twisted Chevalley groups over commutative rings, *Tokyo Kyoiku Daigaku (A)* **13** (1977), 194–218.
2. J. L. ALPERIN AND D. GORENSTEIN, The multiplicators of certain simple groups, *Proc. Amer. Math. Soc.* **17**(1966), 515–519.
3. E. ARTIN, "Geometric Algebra," Interscience, New York, 1957.
4. C. CHEVALLEY, "Theory of Lie Groups," Princeton Univ. Press, Princeton, N. J., 1946.
5. C. CHEVALLEY, "The Algebraic Theory of Spinors," Columbia Univ. Press, New York, 1954.
6. C. CHEVALLEY,Sur certains groupes simples, *Tôkoku Math. J.* **7**(1955), 14–66.
7. V. V. DEODHAR, On central extensions of rational points of algebraic groups, *Amer. J. Math.* **100**(1978), 303–386.
8. R. L. GRIESS, Schur multipliers of the known finite simple groups, *Bull. Amer. Math. Soc.* **78** (1972), 68–71.
9. R. L. GRIESS, Schur multipliers of finite simple groups of Lie type, *Trans. Amer. Math. Soc.* 183 (1973), 355–421.
10. J. GROVER,Covering groups of groups of Lie type, *Pacific J. Math.* **30**(1969), 645–655.
11. I. SCHUR, Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.* **127** (1904), 20–50; **132** (1907), 85–137.
12. I. SCHUR, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.* **139** (1911), 155–250.
13. R. STEINBERG, Générateurs, relations, et revêtements de groupes algébriques, *in* "Colloq. Théorie des Groupes Algébriques, Bruxelles (1962)," pp. 113–127.
14. R. STEINBERG, "Lectures on Chevalley Groups," Yale Univ. Lecture Notes, New Haven Conn., 1967.
15. H. J. ZASSENHAUS, "The Theory of Groups," 2nd ed., Chelsea, New York, 1958.