Birational R-matrix Formulas

Sunita Chepuri

University of Minnesota Combinatorics and Algebra REU

June 16, 2020

Outline

- Totally nonnegative matrices

Outline

- Totally nonnegative matrices
- Planar networks

Outline

- Totally nonnegative matrices
- Planar networks
- Factorization of TNN matrices

Outline

- Totally nonnegative matrices
- Planar networks
- Factorization of TNN matrices
- Cylindric networks

Outline

- Totally nonnegative matrices
- Planar networks
- Factorization of TNN matrices
- Cylindric networks
- Factorization of $U_{\geq 0} \setminus U_{>0}$

Outline

- Totally nonnegative matrices
- Planar networks
- Factorization of TNN matrices
- Cylindric networks
- Factorization of $U_{\geq 0} \setminus U_{>0}$
- Birational R-matrix

Outline

- Totally nonnegative matrices
- Planar networks
- Factorization of TNN matrices
- Cylindric networks
- Factorization of $U_{\geq 0} \setminus U_{>0}$
- Birational R-matrix

Minors

Definition

Let M be an $n \times n$ matrix. A *minor* of M is $\Delta(M)_{I,J} := \det(M_{I,J})$ where $M_{I,J}$ is the submatrix of all entries of M in a row indexed by I and a column indexed by J.

Minors

Definition

Let M be an $n \times n$ matrix. A *minor* of M is $\Delta(M)_{I,J} := \det(M_{I,J})$ where $M_{I,J}$ is the submatrix of all entries of M in a row indexed by I and a column indexed by J.

Example:

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

Minors

Definition

Let M be an $n \times n$ matrix. A *minor* of M is $\Delta(M)_{I,J} := \det(M_{I,J})$ where $M_{I,J}$ is the submatrix of all entries of M in a row indexed by I and a column indexed by J.

Example:

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Delta(M)_{\{1,3\},\{1,2\}} =$$

Minors

Definition

Let M be an $n \times n$ matrix. A *minor* of M is $\Delta(M)_{I,J} := \det(M_{I,J})$ where $M_{I,J}$ is the submatrix of all entries of M in a row indexed by I and a column indexed by J.

Example:

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Delta(M)_{\{1,3\},\{1,2\}} = \det \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = 0.$$

Total Positivity

Definition

A matrix is *totally positive* if $\det(M_{I,J}) > 0$ for any I, J of the same size.
A matrix is *totally nonnegative* if $\det(M_{I,J}) \geq 0$ for any I, J of the same size.

Total Positivity

Definition

A matrix is *totally positive* if $\det(M_{I,J}) > 0$ for any I, J of the same size.
A matrix is *totally nonnegative* if $\det(M_{I,J}) \geq 0$ for any I, J of the same size.

Example:

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

Total Positivity

Definition

A matrix is *totally positive* if $\det(M_{I,J}) > 0$ for any I, J of the same size. A matrix is *totally nonnegative* if $\det(M_{I,J}) \geq 0$ for any I, J of the same size.

Example:

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

M is not totally positive, as $\Delta(M)_{\{1\},\{3\}} = 0$.

Total Positivity

Definition

A matrix is *totally positive* if $\det(M_{I,J}) > 0$ for any I, J of the same size.
A matrix is *totally nonnegative* if $\det(M_{I,J}) \geq 0$ for any I, J of the same size.

Example:

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

M is not totally positive, as $\Delta(M)_{\{1\},\{3\}} = 0$.

We can see all the entries of M are nonnegative.

Total Positivity

Definition

A matrix is *totally positive* if $\det(M_{I,J}) > 0$ for any I, J of the same size.
A matrix is *totally nonnegative* if $\det(M_{I,J}) \geq 0$ for any I, J of the same size.

Example:

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

M is not totally positive, as $\Delta(M)_{\{1\},\{3\}} = 0$.

We can see all the entries of M are nonnegative.

$\Delta(M)_{\{1,3\},\{1,2\}} = 0$.

Total Positivity

Definition

A matrix is *totally positive* if $\det(M_{I,J}) > 0$ for any I, J of the same size. A matrix is *totally nonnegative* if $\det(M_{I,J}) \geq 0$ for any I, J of the same size.

Example:

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

M is not totally positive, as $\Delta(M)_{\{1\},\{3\}} = 0$.

We can see all the entries of M are nonnegative.

$\Delta(M)_{\{1,3\},\{1,2\}} = 0$.

$\Delta(M)_{\{1,2,3\},\{1,2,3\}} = \det(M) = 24$.

Total Positivity

Definition

A matrix is *totally positive* if $\det(M_{I,J}) > 0$ for any I, J of the same size. A matrix is *totally nonnegative* if $\det(M_{I,J}) \geq 0$ for any I, J of the same size.

Example:

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

M is not totally positive, as $\Delta(M)_{\{1\},\{3\}} = 0$.

We can see all the entries of M are nonnegative.

$\Delta(M)_{\{1,3\},\{1,2\}} = 0$.

$\Delta(M)_{\{1,2,3\},\{1,2,3\}} = \det(M) = 24$.

We could continue checking and see that M is totally nonnegative.

Motivation

Why is this interesting?

Motivation

Why is this interesting?

- Have variation diminishing property

Motivation

Why is this interesting?

- Have variation diminishing property
- Nice eigenvalues

Motivation

Why is this interesting?

- Have variation diminishing property
- Nice eigenvalues
- Can be used in functional analysis, ODE's, probability, statistics

Motivation

Why is this interesting?

- Have variation diminishing property
- Nice eigenvalues
- Can be used in functional analysis, ODE's, probability, statistics
- Relates combinatorially to networks

Motivation

Why is this interesting?

- Have variation diminishing property
- Nice eigenvalues
- Can be used in functional analysis, ODE's, probability, statistics
- Relates combinatorially to networks
- Led to development of cluster algebras

Outline

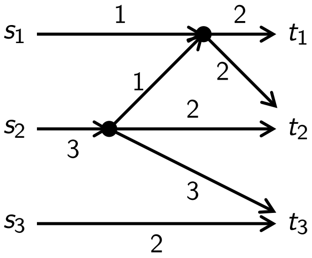
- Totally nonnegative matrices
- Planar networks
- Factorization of TNN matrices
- Cylindric networks
- Factorization of $U_{\geq 0} \setminus U_{>0}$
- Birational R-matrix

Planar Networks

We will be considering planar, directed, acyclic, edge-weighted graphs with n sources and n sinks, where the sources and sinks are separated. We will call these *planar networks*.

Planar Networks

We will be considering planar, directed, acyclic, edge-weighted graphs with n sources and n sinks, where the sources and sinks are separated. We will call these *planar networks*.



Path Matrix

Definition

For any directed path from a source to a sink, the *weight* of the path is the product of weights of the edges.

Path Matrix

Definition

For any directed path from a source to a sink, the *weight* of the path is the product of weights of the edges.

The *path matrix* is the matrix $M = (m_{i,j})$ where

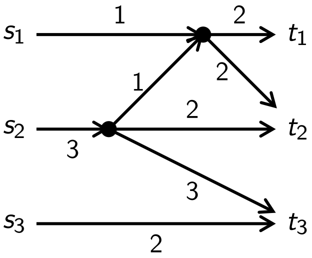
$$m_{i,j} = \sum_{\text{paths } P: s_i \to t_j} wt(P).$$

Path Matrix

Definition

For any directed path from a source to a sink, the *weight* of the path is the product of weights of the edges.
The *path matrix* is the matrix $M = (m_{i,j})$ where

$$m_{i,j} = \sum_{\text{paths } P: s_i \to t_j} wt(P).$$



$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

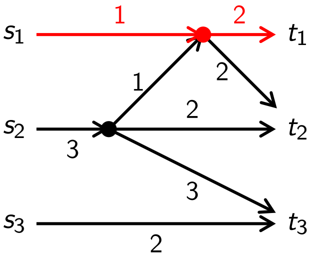
Path Matrix

Definition

For any directed path from a source to a sink, the *weight* of the path is the product of weights of the edges.

The *path matrix* is the matrix $M = (m_{i,j})$ where

$$m_{i,j} = \sum_{\text{paths } P: s_i \to t_j} wt(P).$$



$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

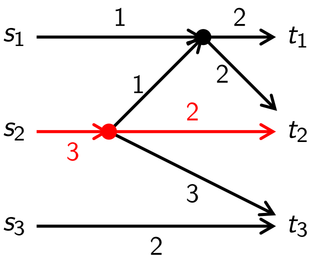
$$m_{11} = (1)(2)$$

Path Matrix

Definition

For any directed path from a source to a sink, the *weight* of the path is the product of weights of the edges.
The *path matrix* is the matrix $M = (m_{i,j})$ where

$$m_{i,j} = \sum_{\text{paths } P: s_i \to t_j} wt(P).$$



$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

$$m_{11} = (1)(2)$$

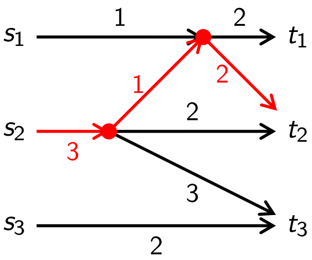
$$m_{22} = (3)(2) + (3)(1)(2)$$

Path Matrix

Definition

For any directed path from a source to a sink, the *weight* of the path is the product of weights of the edges.
The *path matrix* is the matrix $M = (m_{i,j})$ where

$$m_{i,j} = \sum_{\text{paths } P: s_i \to t_j} wt(P).$$



$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

$$m_{11} = (1)(2)$$

$$m_{22} = (3)(2) + (3)(1)(2)$$

Lindström's Lemma

Theorem (Lindström's Lemma, 1973)

The weight matrix of a planar network is a totally nonnegative matrix. In particular,

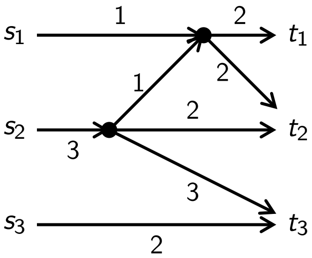
$$\det(M_{I,J}) = \sum_{\substack{\text{families of nonintersecting} \\ \text{paths from sources indexed} \\ \text{by } I \text{ to sinks indexed by } J}} \left(\prod_{\substack{\text{all paths } P \\ \text{in a family}}} wt(P) \right).$$

Lindström's Lemma

Theorem (Lindström's Lemma, 1973)

The weight matrix of a planar network is a totally nonnegative matrix. In particular,

$$\det(M_{I,J}) = \sum_{\substack{\text{families of nonintersecting} \\ \text{paths from sources indexed} \\ \text{by } I \text{ to sinks indexed by } J}} \left(\prod_{\substack{\text{all paths } P \\ \text{in a family}}} wt(P) \right).$$



$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

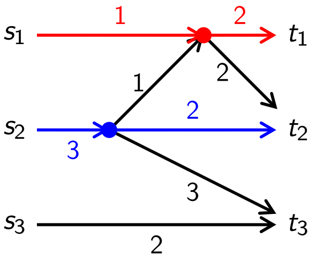
$$\det(M_{\{1,2\},\{1,2\}}) = 12$$

Lindström's Lemma

Theorem (Lindström's Lemma, 1973)

The weight matrix of a planar network is a totally nonnegative matrix. In particular,

$$\det(M_{I,J}) = \sum_{\substack{\text{families of nonintersecting} \\ \text{paths from sources indexed} \\ \text{by } I \text{ to sinks indexed by } J}} \left(\prod_{\substack{\text{all paths } P \\ \text{in a family}}} wt(P) \right).$$



$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

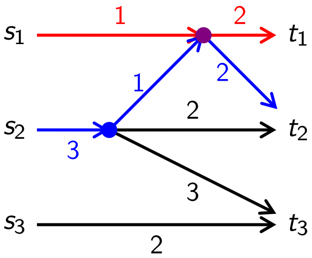
$$\det(M_{\{1,2\},\{1,2\}}) = 12$$

Lindström's Lemma

Theorem (Lindström's Lemma, 1973)

The weight matrix of a planar network is a totally nonnegative matrix. In particular,

$$\det(M_{I,J}) = \sum_{\substack{\text{families of nonintersecting} \\ \text{paths from sources indexed} \\ \text{by } I \text{ to sinks indexed by } J}} \left(\prod_{\substack{\text{all paths } P \\ \text{in a family}}} wt(P) \right).$$



$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

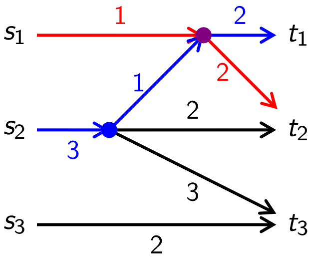
$$\det(M_{\{1,2\},\{1,2\}}) = 12$$

Lindström's Lemma

Theorem (Lindström's Lemma, 1973)

The weight matrix of a planar network is a totally nonnegative matrix. In particular,

$$\det(M_{I,J}) = \sum_{\substack{\text{families of nonintersecting} \\ \text{paths from sources indexed} \\ \text{by } I \text{ to sinks indexed by } J}} \left(\prod_{\substack{\text{all paths } P \\ \text{in a family}}} wt(P) \right).$$



$$M = \begin{bmatrix} 2 & 2 & 0 \\ 6 & 12 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(M_{\{1,2\},\{1,2\}}) = 12$$

The converse is also true...

Lindström's Lemma

The converse is also true...

Theorem (Brenti, 1995)

Every nonnegative matrix is the weight matrix of a planar network.

Lindström's Lemma

The converse is also true...

Theorem (Brenti, 1995)

Every nonnegative matrix is the weight matrix of a planar network.

For elements of $GL_n(\mathbb{R})$, one way to prove this is by considering factorizations of totally nonnegative matrices.

Outline

- Totally nonnegative matrices
- Planar networks
- Factorization of TNN matrices
- Cylindric networks
- Factorization of $U_{\geq 0} \setminus U_{>0}$
- Birational R-matrix

Loewner-Whitney Theorem

Loewner-Whitney Theorem

Any invertible totally nonnegative matrix is a product of elementary Jacobi matrices with nonnegative matrix entries.

Loewner-Whitney Theorem

Loewner-Whitney Theorem

Any invertible totally nonnegative matrix is a product of elementary Jacobi matrices with nonnegative matrix entries.

Elementary Jacobi Matrices:

Loewner-Whitney Theorem

Loewner-Whitney Theorem

Any invertible totally nonnegative matrix is a product of elementary Jacobi matrices with nonnegative matrix entries.

Elementary Jacobi Matrices:

- $e_i(a)$ is the identity but with a in the $i, i+1$ entry.

Loewner-Whitney Theorem

Loewner-Whitney Theorem

Any invertible totally nonnegative matrix is a product of elementary Jacobi matrices with nonnegative matrix entries.

Elementary Jacobi Matrices:

- $e_i(a)$ is the identity but with a in the $i, i+1$ entry.
- $f_i(a)$ is the identity but with a in the $i+1, i$ entry ($f_i(a) = e_i(a)^T$).

Loewner-Whitney Theorem

Any invertible totally nonnegative matrix is a product of elementary Jacobi matrices with nonnegative matrix entries.

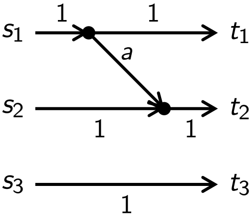
Elementary Jacobi Matrices:

- $e_i(a)$ is the identity but with a in the $i, i+1$ entry.
- $f_i(a)$ is the identity but with a in the $i+1, i$ entry ($f_i(a) = e_i(a)^T$).
- $h_i(a)$ is the identity but with a in the i, i entry.

Example: $n = 3$

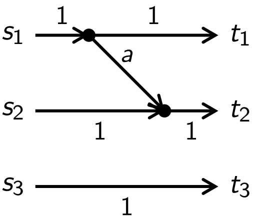
$$e_1(a) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad f_2(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix} \quad h_3(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

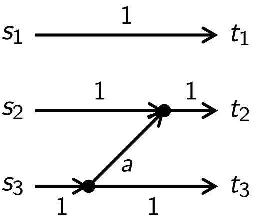
Networks for Elementary Jacobi Matrices



$$e_1(a) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

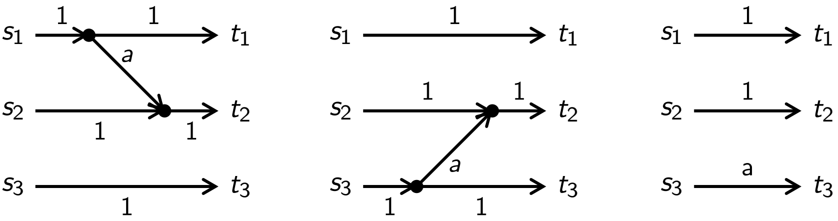
Networks for Elementary Jacobi Matrices





$$e_1(a) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad f_2(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$$

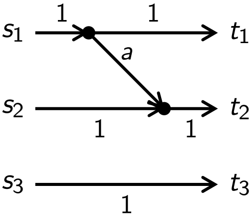
Networks for Elementary Jacobi Matrices



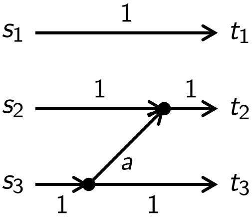
$$e_1(a) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad f_2(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix} \qquad h_3(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

Concatenation of networks is multiplication of matrices.

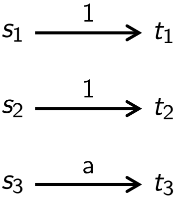
Networks for Elementary Jacobi Matrices



$$e_1(a) = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$f_2(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$$



$$h_3(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

Concatenation of networks is multiplication of matrices.

This proves that every element of $GL_n(\mathbb{R})_{\geq 0}$ is the weight matrix of a planar network.

Outline

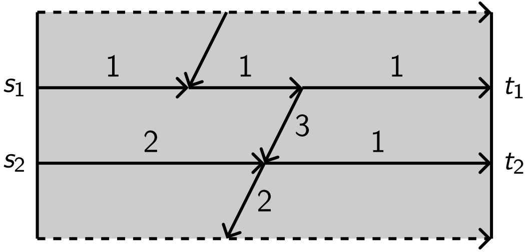
- Totally nonnegative matrices
- Planar networks
- Factorization of TNN matrices
- Cylindric networks
- Factorization of $U_{\geq 0} \setminus U_{>0}$
- Birational R-matrix

Cylindric Planar Networks

Let's loosen the planarity condition of our planar networks by embedding them in a cylinder.

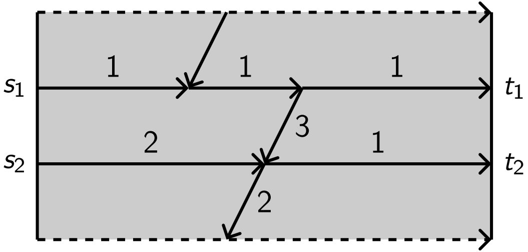
Cylindric Planar Networks

Let's loosen the planarity condition of our planar networks by embedding them in a cylinder.



Cylindric Planar Networks

Let's loosen the planarity condition of our planar networks by embedding them in a cylinder.



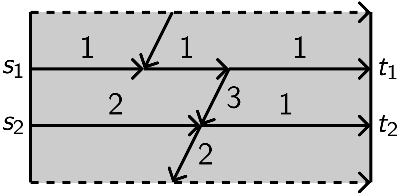
We'll have a *chord* $\mathfrak{h}$ from the left boundary component to the right, n sources on the left labeled from top to bottom, and n sinks on the right labeled from top to bottom.

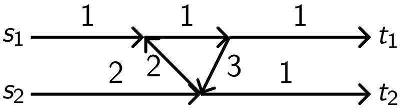
Cylindric Planar Networks

We want our network to be acyclic in the sense that there are no cycles in the network when drawn on the universal cover of the cylinder.

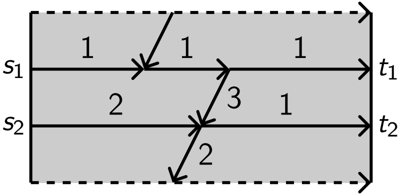
Cylindric Planar Networks

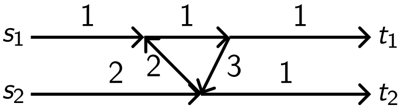
We want our network to be acyclic in the sense that there are no cycles in the network when drawn on the universal cover of the cylinder.

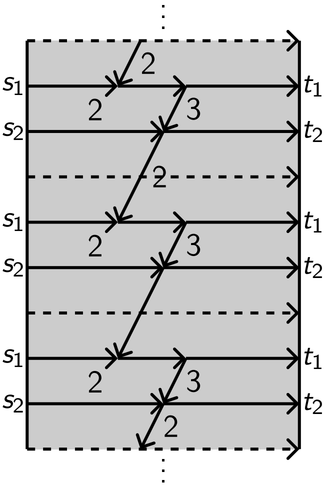




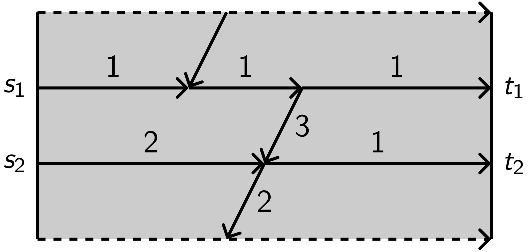
We want our network to be acyclic in the sense that there are no cycles in the network when drawn on the universal cover of the cylinder.







Weight Matrix for Cylindric Network



$$\begin{bmatrix} 1 + 6t + 36t^2 + \dots & 3 + 18t + 108t^2 \\ 4t + 24t^2 + \dots & 2 + 12t + 72t^2 + \dots \end{bmatrix}$$

Unfolding

Given a matrix where the entries are formal series in t, t^{-1}, we can define the *unfolding* of the matrix.

Unfolding

Given a matrix where the entries are formal series in t, t^{-1}, we can define the *unfolding* of the matrix.

$$\begin{bmatrix} 1 + 6t + 36t^2 + ... & 3 + 18t + 108t^2 \\ 4t + 24t^2 + ... & 2 + 12t + 72t^2 + ... \end{bmatrix}$$

Unfolding

Given a matrix where the entries are formal series in t, t^{-1}, we can define the *unfolding* of the matrix.

$$\begin{bmatrix} 1 + 6t + 36t^2 + \ldots & 3 + 18t + 108t^2 \\ 4t + 24t^2 + \ldots & 2 + 12t + 72t^2 + \ldots \end{bmatrix}$$

$$\begin{bmatrix} \ddots & & & & & & & \iddots \\ & 1 & 3 & 6 & 18 & 36 & 108 & \\ & 0 & 2 & 4 & 12 & 24 & 72 & \\ & 0 & 0 & 1 & 3 & 6 & 18 & \\ & 0 & 0 & 0 & 2 & 4 & 12 & \\ \iddots & & & & & & & \ddots \end{bmatrix}$$

Cylindric Lindström's Lemma

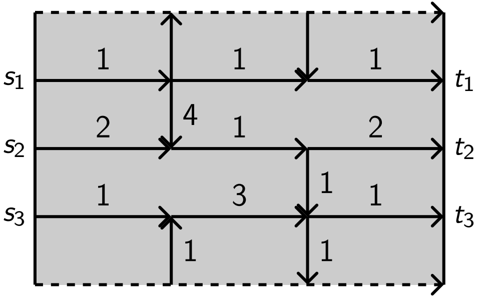
Cylindric Lindström Lemma, Lam–Pylyavskyy 2008

The unfolding of weight matrix of a cylindric network N is totally nonnegative. In particular,

$$\det(M_{I,J}) = \sum_{\substack{\text{families of nonintersecting} \\ \text{paths from sources indexed} \\ \text{by } I \text{ to sinks indexed by } J \\ \text{in the universal cover of } N}} \left(\prod_{\substack{\text{all paths } P \\ \text{in a family}}} wt(P) \right).$$

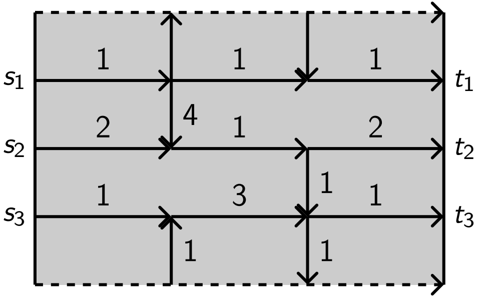
REU Exercise

Consider the following cylindric network N:



REU Exercise

Consider the following cylindric network N:

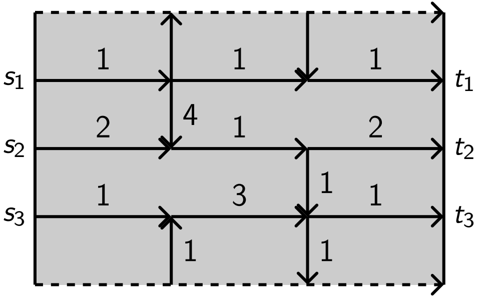


REU Exercise 2.1

a. Compute the weight matrix of N.

REU Exercise

Consider the following cylindric network N:

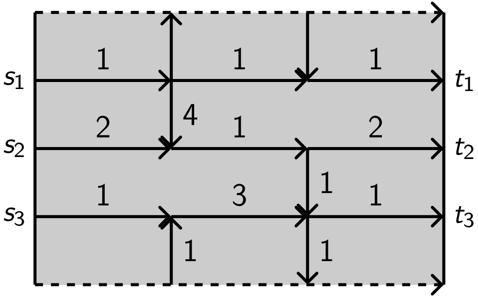


REU Exercise 2.1

- (a) Compute the weight matrix of N.
- (b) Compute the unfolding of the path matrix of N.

REU Exercise

Consider the following cylindric network N:



REU Exercise 2.1

1. (a) Compute the weight matrix of N.
2. (b) Compute the unfolding of the path matrix of N.
3. (c) Check that the Cylindric Lindström Lemma holds for N.

Restriction to $U_{\geq 0} \setminus U_{>0}$

In general, we can't go the other direction. One barrier - a totally nonnegative infinite periodic matrix might require an infinite network to construct it.

Restriction to $U_{\geq 0} \setminus U_{>0}$

In general, we can't go the other direction. One barrier - a totally nonnegative infinite periodic matrix might require an infinite network to construct it.

Definition

Let $GL_n(\mathbb{R}((t)))$ be the set of $n \times n$ matrices with entries that are formal Laurent series in t and that have a nonzero determinant.

Restriction to $U_{\geq 0} \setminus U_{>0}$

In general, we can't go the other direction. One barrier - a totally nonnegative infinite periodic matrix might require an infinite network to construct it.

Definition

Let $GL_n(\mathbb{R}((t)))$ be the set of $n \times n$ matrices with entries that are formal Laurent series in t and that have a nonzero determinant. Let $U \subseteq GL_n(\mathbb{R}((t)))$ be the elements of $GL_n(\mathbb{R}((t)))$ with upper unitriangular unfoldings.

Restriction to $U_{\geq 0} \setminus U_{>0}$

In general, we can't go the other direction. One barrier - a totally nonnegative infinite periodic matrix might require an infinite network to construct it.

Definition

Let $GL_n(\mathbb{R}((t)))$ be the set of $n \times n$ matrices with entries that are formal Laurent series in t and that have a nonzero determinant. Let $U \subseteq GL_n(\mathbb{R}((t)))$ be the elements of $GL_n(\mathbb{R}((t)))$ with upper unitriangular unfoldings.

Define $U_{\geq 0} \subseteq U$ as the elements of U with totally nonnegative unfoldings and $U_{>0} \subseteq U$ as the elements of U with totally positive unfoldings in the sense that all minors that are not forced to be 0 are positive.

Restriction to $U_{\geq 0} \setminus U_{>0}$

Theorem, Lam–Pylyavskyy 2008

Every element of $U_{\geq 0} \setminus U_{>0}$ is the weight matrix of a cylindric network.

Restriction to $U_{\geq 0} \setminus U_{>0}$

Theorem, Lam–Pylyavskyy 2008

Every element of $U_{\geq 0} \setminus U_{>0}$ is the weight matrix of a cylindric network.

Again, we can prove this by factorizations.

Outline

- Totally nonnegative matrices
- Planar networks
- Factorization of TNN matrices
- Cylindric networks
- Factorization of $U_{\geq 0} \setminus U_{>0}$
- Birational R-matrix

Whirls and Curls

Whirls: $M(a_1, a_2, a_3) = \begin{bmatrix} 1 & a_1 & 0 \\ 0 & 1 & a_2 \\ a_3 t & 0 & 1 \end{bmatrix}$

Whirls and Curls

Whirls: $M(a_1, a_2, a_3) = \begin{bmatrix} 1 & a_1 & 0 \\ 0 & 1 & a_2 \\ a_3 t & 0 & 1 \end{bmatrix}$

Curls: $N(a_1, a_2, a_3) = \left(\sum_{k=0}^{\infty} (a_1 a_2 a_3 t)^k\right) \begin{bmatrix} 1 & a_1 & a_1 a_2 \\ a_2 a_3 t & 1 & a_2 \\ a_3 t & a_1 a_3 t & 1 \end{bmatrix}$

Whirls and Curls

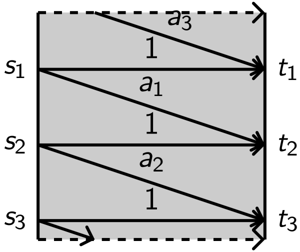
Whirls: $M(a_1, a_2, a_3) = \begin{bmatrix} 1 & a_1 & 0 \\ 0 & 1 & a_2 \\ a_3 t & 0 & 1 \end{bmatrix}$

Curls: $N(a_1, a_2, a_3) = \left(\sum_{k=0}^{\infty} (a_1 a_2 a_3 t)^k\right) \begin{bmatrix} 1 & a_1 & a_1 a_2 \\ a_2 a_3 t & 1 & a_2 \\ a_3 t & a_1 a_3 t & 1 \end{bmatrix}$

Theorem, Lam–Pylyavskyy 2008

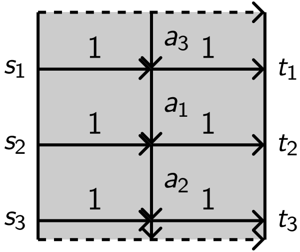
Any element of $U_{\geq 0} \setminus U_{>0}$ is a product of whirls and curls with nonnegative parameters.

Networks for Whirls



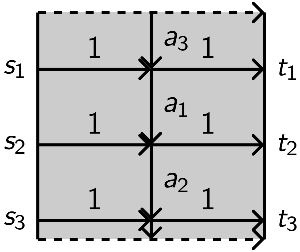
$$M(a_1, a_2, a_3) = \begin{bmatrix} 1 & a_1 & 0 \\ 0 & 1 & a_2 \\ a_3 t & 0 & 1 \end{bmatrix}$$

Network for Curls



$$N(a_1, a_2, a_3) = \left(\sum_{k=0}^{\infty}(a_1 a_2 a_3 t)^k\right) \begin{bmatrix} 1 & a_1 & a_1 a_2 \\ a_2 a_3 t & 1 & a_2 \\ a_3 t & a_1 a_3 t & 1 \end{bmatrix}$$

Network for Curls



$$N(a_1, a_2, a_3) = \left(\sum_{k=0}^{\infty}(a_1 a_2 a_3 t)^k\right) \begin{bmatrix} 1 & a_1 & a_1 a_2 \\ a_2 a_3 t & 1 & a_2 \\ a_3 t & a_1 a_3 t & 1 \end{bmatrix}$$

Since concatenation of networks is multiplication of matrices, this proves that every element of $U_{\geq 0} \setminus U_{>0}$ is the weight matrix of a cylindric network.

Inverse Problems

Inverse Problems: Given the boundary measurements of a system, can we recover its interior parameters?

Inverse Problems

Inverse Problems: Given the boundary measurements of a system, can we recover its interior parameters?

In our case: Given an element of $GL_n(\mathbb{R})$ or $U_{\geq 0} \setminus U_{>0}$ and a factorization, can we recover the parameters of the factorization?

Example for $GL_n(\mathbb{R})$

We will restrict to the set of upper unitriangular matrices if $GL_n(\mathbb{R})$.

Example for $GL_n(\mathbb{R})$

We will restrict to the set of upper unitriangular matrices if $GL_n(\mathbb{R})$.

Let M be a totally nonnegative upper unitriangular element of $GL_n(\mathbb{R})$. Then M can be factored into a product of e_i's.

Example for $GL_n(\mathbb{R})$

We will restrict to the set of upper unitriangular matrices if $GL_n(\mathbb{R})$.

Let M be a totally nonnegative upper unitriangular element of $GL_n(\mathbb{R})$. Then M can be factored into a product of e_i's.

Given a shortest factorization, there is a unique way to determine the parameters (Berenstein–Fomin–Zelevinsky 1996).

Example for $GL_n(\mathbb{R})$

We will restrict to the set of upper unitriangular matrices if $GL_n(\mathbb{R})$.

Let M be a totally nonnegative upper unitriangular element of $GL_n(\mathbb{R})$. Then M can be factored into a product of e_i's.

Given a shortest factorization, there is a unique way to determine the parameters (Berenstein–Fomin–Zelevinsky 1996).

Example:

$$\begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = e_1(2)e_2(1)e_1(5)$$

$$= e_2\left(\frac{5}{7}\right) e_1(7) e_2\left(\frac{2}{7}\right)$$

Birational R-Matrix

Again, things aren't as nice in our other case.

Again, things aren't as nice in our other case.

Example:

$$\begin{bmatrix} 1 & 3 & 1 \\ 4t & 1 & 3 \\ 3t & 2t & 1 \end{bmatrix} = M(1,2,1)M(2,1,2)$$

$$= M\left(\frac{7}{5}, \frac{16}{7}, \frac{5}{4}\right) M\left(\frac{8}{5}, \frac{5}{7}, \frac{7}{4}\right)$$

$$\begin{bmatrix} 6t+1 & 2t+3 & 8 \\ 5t & 3t+1 & 4t+3 \\ 8t^2+3t & 7t & 12t+1 \end{bmatrix} = N(2,1,2)N(1,2,1)$$

$$= N\left(\frac{8}{5}, \frac{5}{7}, \frac{7}{4}\right) N\left(\frac{7}{5}, \frac{16}{7}, \frac{5}{4}\right)$$

Birational R-Matrix

Definition

Let $\mathbf{a} = (a_1, ..., a_n), \mathbf{b} = (b_1, ..., b_n) \in \mathbb{R}^n_{\geq 0}$. Let

$$\kappa_i(\mathbf{a}, \mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k.$$

Define η as the map that sends $(\mathbf{a}, \mathbf{b})$ to $(\mathbf{b}', \mathbf{a}')$ where

$$b'_i = \frac{b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})} \qquad a'_i = \frac{a_{i-1}\kappa_{i-1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}.$$

Birational R-Matrix

Definition

Let $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n_{\geq 0}$. Let

$$\kappa_i(\mathbf{a}, \mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k.$$

Define η as the map that sends $(\mathbf{a}, \mathbf{b})$ to $(\mathbf{b}', \mathbf{a}')$ where

$$b'_i = \frac{b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})} \qquad a'_i = \frac{a_{i-1}\kappa_{i-1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}.$$

Example: $\mathbf{a} = (1, 2, 1), \mathbf{b} = (2, 1, 2)$.

Birational R-Matrix

Definition

Let $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n_{\geq 0}$. Let

$$\kappa_i(\mathbf{a}, \mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k.$$

Define η as the map that sends $(\mathbf{a}, \mathbf{b})$ to $(\mathbf{b}', \mathbf{a}')$ where

$$b'_i = \frac{b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})} \qquad a'_i = \frac{a_{i-1}\kappa_{i-1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}.$$

Example: $\mathbf{a} = (1, 2, 1), \mathbf{b} = (2, 1, 2)$.

$\kappa_1(\mathbf{a}, \mathbf{b}) = a_2 a_3 + b_2 a_3 + b_2 b_3 = 2 + 1 + 2 = 5$

Birational R-Matrix

Definition

Let $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n_{\geq 0}$. Let

$$\kappa_i(\mathbf{a}, \mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k.$$

Define η as the map that sends $(\mathbf{a}, \mathbf{b})$ to $(\mathbf{b}', \mathbf{a}')$ where

$$b'_i = \frac{b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})} \qquad a'_i = \frac{a_{i-1}\kappa_{i-1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}.$$

Example: $\mathbf{a} = (1, 2, 1), \mathbf{b} = (2, 1, 2)$.

$\kappa_1(\mathbf{a}, \mathbf{b}) = a_2 a_3 + b_2 a_3 + b_2 b_3 = 2 + 1 + 2 = 5$

$\kappa_2(\mathbf{a}, \mathbf{b}) = a_1 a_3 + a_1 b_3 + b_1 b_3 = 1 + 2 + 4 = 7$

Birational R-Matrix

Definition

Let $\mathbf{a} = (a_1, ..., a_n), \mathbf{b} = (b_1, ..., b_n) \in \mathbb{R}^n_{\geq 0}$. Let

$$\kappa_i(\mathbf{a}, \mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k.$$

Define η as the map that sends $(\mathbf{a}, \mathbf{b})$ to $(\mathbf{b}', \mathbf{a}')$ where

$$b'_i = \frac{b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})} \qquad a'_i = \frac{a_{i-1}\kappa_{i-1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}.$$

Example: $\mathbf{a} = (1, 2, 1), \mathbf{b} = (2, 1, 2)$.

$\kappa_1(\mathbf{a}, \mathbf{b}) = a_2a_3 + b_2a_3 + b_2b_3 = 2 + 1 + 2 = 5$

$\kappa_2(\mathbf{a}, \mathbf{b}) = a_1a_3 + a_1b_3 + b_1b_3 = 1 + 2 + 4 = 7$

$b_1 = \frac{b_2\kappa_2(\mathbf{a},\mathbf{b})}{\kappa_1(\mathbf{a},\mathbf{b})} = \frac{7}{5}$

Birational R-Matrix

Theorem, Lam–Pylyavskyy 2008

The birational R-matrix has the following properties:

Birational R-Matrix

Theorem, Lam–Pylyavskyy 2008

The birational R-matrix has the following properties:

1. $M(\mathbf{a})M(\mathbf{b}) = M(\mathbf{b}')M(\mathbf{a}')$ and $N(\mathbf{b})N(\mathbf{a}) = N(\mathbf{a}')N(\mathbf{b}')$.

Birational R-Matrix

Theorem, Lam–Pylyavskyy 2008

The birational R-matrix has the following properties:

1. $M(\mathbf{a})M(\mathbf{b}) = M(\mathbf{b}')M(\mathbf{a}')$ and $N(\mathbf{b})N(\mathbf{a}) = N(\mathbf{a}')N(\mathbf{b}')$.
2. η is an involution ($\eta^2 = 1$).

Birational R-Matrix

Theorem, Lam–Pylyavskyy 2008

The birational R-matrix has the following properties:

1. $M(\mathbf{a})M(\mathbf{b}) = M(\mathbf{b}')M(\mathbf{a}')$ and $N(\mathbf{b})N(\mathbf{a}) = N(\mathbf{a}')N(\mathbf{b}')$.
2. η is an involution ($\eta^2 = 1$).
3. For $1 \leq i < k$,
$\eta_i \circ \eta_{i+1} \circ \eta_i(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, ..., \mathbf{a}^{(k)}) = \eta_{i+1} \circ \eta_i \circ \eta_{i+1}(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, ..., \mathbf{a}^{(k)})$

Birational R-Matrix

Theorem, Lam–Pylyavskyy 2008

The birational R-matrix has the following properties:

1. $M(\mathbf{a})M(\mathbf{b}) = M(\mathbf{b}')M(\mathbf{a}')$ and $N(\mathbf{b})N(\mathbf{a}) = N(\mathbf{a}')N(\mathbf{b}')$.
2. η is an involution ($\eta^2 = 1$).
3. For $1 \leq i < k$,
   $\eta_i \circ \eta_{i+1} \circ \eta_i(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)}) = \eta_{i+1} \circ \eta_i \circ \eta_{i+1}(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)})$

Note that the last two properties implies that η gives an action of the symmetric group on whirls/curls in a matrix factorization.

REU Exercises

REU Exercise 2.2

a. Compute $\eta(\mathbf{a}, \mathbf{b})$ when $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (2, 3, 4)$.

REU Exercises

REU Exercise 2.2

1. Compute $\eta(\mathbf{a}, \mathbf{b})$ when $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (2, 3, 4)$.
2. Verify that (1) from the previous slide holds in this case.

REU Exercise 2.2

a. Compute $\eta(\mathbf{a}, \mathbf{b})$ when $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (2, 3, 4)$.
b. Verify that (1) from the previous slide holds in this case.

REU Exercise 2.3

Verify that all three properties hold when $n = 2$.

REU Problem

REU Problem 2

The birational R-matrix formula is a formula for how transpositions act on factorizations. Find a (combinatorial) formula for how the other elements of the symmetric group act.

REU Problem

REU Problem 2

The birational R-matrix formula is a formula for how transpositions act on factorizations. Find a (combinatorial) formula for how the other elements of the symmetric group act.

Example: $A = M(\mathbf{a})M(\mathbf{b})M(\mathbf{c})$

REU Problem

REU Problem 2

The birational R-matrix formula is a formula for how transpositions act on factorizations. Find a (combinatorial) formula for how the other elements of the symmetric group act.

Example: $A = M(\mathbf{a})M(\mathbf{b})M(\mathbf{c})$

If $\eta(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$, then $A = M(\mathbf{b}')M(\mathbf{a}')M(\mathbf{c})$ (think of this as the action of (12)).

REU Problem

REU Problem 2

The birational R-matrix formula is a formula for how transpositions act on factorizations. Find a (combinatorial) formula for how the other elements of the symmetric group act.

Example: $A = M(\mathbf{a})M(\mathbf{b})M(\mathbf{c})$

If $\eta(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$, then $A = M(\mathbf{b}')M(\mathbf{a}')M(\mathbf{c})$ (think of this as the action of (12)).

If $\eta(\mathbf{a}', \mathbf{c}) = (\mathbf{a}'', \mathbf{c}'')$, then $A = M(\mathbf{b}')M(\mathbf{c}'')M(\mathbf{a}'')$ (think of this as the action of (132)).

REU Problem

REU Problem 2

The birational R-matrix formula is a formula for how transpositions act on factorizations. Find a (combinatorial) formula for how the other elements of the symmetric group act.

Example: $A = M(\mathbf{a})M(\mathbf{b})M(\mathbf{c})$

If $\eta(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$, then $A = M(\mathbf{b}')M(\mathbf{a}')M(\mathbf{c})$ (think of this as the action of (12)).

If $\eta(\mathbf{a}', \mathbf{c}) = (\mathbf{a}'', \mathbf{c}'')$, then $A = M(\mathbf{b}')M(\mathbf{c}'')M(\mathbf{a}'')$ (think of this as the action of (132)).

What are the formulas for $\mathbf{a}''$ and $\mathbf{c}''$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$?

Exercise

REU Exercise 2.4

For $n = 2$ and $n = 3$, compute formulas for the actions of (123), (132), and (13).

Exercise

REU Exercise 2.4

For $n = 2$ and $n = 3$, compute formulas for the actions of (123), (132), and (13).

Note: This might best be done using software.

References

- T. Lam and P. Pylyavskyy, *Total positivity in loop groups, I: Whirls and curls*, Advances in Mathematics, **230** (2012), no. 3, 1222–1271.
- T. Lam, *Loop symmetric functions and factorizing matrix polynomials*, Fifth International Congress of Chinese Mathematicians, **1** (2012), no. 2, 609–627.
- T. Lam and P. Pylyavskyy, *Crystals and total positivity on orientable surfaces*, Selecta Mathematica, **19** (2013), no. 1, 173–235.
- T. Lam and P. Pylyavskyy, *Intrinsic energy is a loop Schur function*, (2010), arXiv:1003.3948.
- A. Berenstein, S. Fomin, and A. Zelevinsky, *Parametrizations of Canonical Bases and Totally Positive Matrices*, Advances in Mathematics, **122** (1996), no. 1, 49–149.