Unimodal, log-concave, and Polya frequency property for graphs on cylinders

UMN
REU Prob 4
2020

Defn A sequence $a_0, a_1, \ldots, a_n$ of real numbers is unimodal if for some $s \in [n]$, $a_0 \le a_1 \le \cdots \le a_s \ge a_{s+1} \ge \cdots \ge a_n$.

Plot the points (i, a_i);

"one peak"

Eg

- $a_k = \binom{n}{k}$ is unimodal in k (for any fixed n)

$\binom{4}{0} < \binom{4}{1} < \binom{4}{2} > \binom{4}{3} > \binom{4}{4}$

1 4 6 4 1

row of Pascal's triangle

2nd Stirling

- $S(n,k) =$ # of set-partitions of $[n]$ in k blocks

unimodal in k (fixed n)

$S(4,1) \leq S(4,2) \geq S(4,3) \geq S(4,4)$

1 7 6 1

1234	1|234 × 4	1|2|34 × 6	1|2|3|4
	12|34 × 3		

1st Stirling

- $c(n,k) =$ # of perms $\in S_n$ with k cycles

unimodal in k

$c(4,1) < c(4,2) > c(4,3) > c(4,4)$

6 11 6 1

(abcd)	(abc) × 8	(ab)	id
	(ab)(cd) × 3		

Kostka number

- Let $K_{\lambda\mu} = $ # of SSYT's of shape λ and weight μ $= [x^\mu]\, S_\lambda(x_1,\dots,x_n)$ $n \gg 0$.

coeff. of x^μ in S_λ

$$\alpha_{ij} = (0,\dots,1,0,\dots,-1,0\dots0).$$

i^{th} spot $\quad$ j^{th} spot

Then $\left(K_{\lambda,\,\mu+k\alpha_{ij}}\right)_{k\in\mathbb{Z}}$ is unimodal.

e.g.: $\lambda = (541)$, $\mu = (433)$, $i,j = 1,2$

$$\dots = K_{\lambda 163} < K_{\lambda,253} < K_{\lambda,343} = K_{\lambda 433} > K_{\lambda,523} > K_{613} = \dots$$

$$\dots \quad 0 \quad 1 \quad 2 \quad 2 \quad 1 \quad 0 \quad \dots$$

$K_{\lambda,253}$: $\begin{array}{l} 11222 \\ 2233 \\ 3 \end{array}$

$K_{\lambda,343}$: $\begin{array}{l} 11122 \\ 2233 \\ 3 \end{array}$ $\quad$ $\begin{array}{l} 11123 \\ 2223 \\ 3 \end{array}$

$K_{\lambda,433}$: $\begin{array}{l} 11112 \\ 2233 \\ 3 \end{array}$ $\quad$ $\begin{array}{l} 11113 \\ 2223 \\ 3 \end{array}$

$K_{\lambda,523}$: $\begin{array}{l} 11111 \\ 2233 \\ 3 \end{array}$

+ many more examples (posets, polytopes, graph theory, rep theory, alg geom)

Defn Seq is *log-concave* if $a_i^2 \geq a_{i-1} a_{i+1}$ $\forall i$.

$\Rightarrow: 2 \log a_i \geq \log a_{i-1} + \log a_{i+1}$

concavity

Rev Ex 4.1 If $(a_i)_{i=0}^n \in \mathbb{R}_{>0}$ is log-concave, then it is unimodal. The same holds if $(a_i)_{i=0}^n \in \mathbb{R}_{\geq 0}$ has no internal zeros: $(\ldots, \neq 0, 0, \neq 0, \ldots)$

"Best" proof of unimodality

Sps $a_i = \# T_i$. Construct maps:

$$T_0 \hookrightarrow T_1 \hookrightarrow \cdots \hookrightarrow T_s \hookleftarrow T_{s+1} \hookleftarrow \cdots \hookleftarrow T_n.$$

(or: switch "$\hookrightarrow$" to "$\twoheadrightarrow$" and reverse direction of maps).

"Best" proof of log-concavity

construct maps $T_{i-1} \times T_{i+1} \hookrightarrow T_i \times T_i$

Defn $(a_i)_{i=0}^{n} \in \mathbb{R}_{\geq 0}$ is a Pólya Frequency Sequence (PFS) if its generating function $\sum_{i=0}^{n} a_i t^i$ has only real roots.

Thm (Edrei, Thoma ~1950s): $(a_i)_{i=0}^{n}$ is a P.F.S. (⟺ real-rooted poly) $\iff$ the infinite matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & \\ & a_0 & a_1 & a_2 & \cdots & a_n & & 0 \\ & & a_0 & a_1 & \cdots & \cdots & a_n & \\ & 0 & & \ddots & \ddots & \ddots & & \ddots \end{pmatrix}$$

is totally nonnegative (all minors are ≥ 0).

Rmk: $\Rightarrow$ not so hard (uses Schur function identities).

$\Leftarrow$ hard.

Rev Ex 4.2

- P.F.S $\Rightarrow$ log-concave with no internal zeros. ($\overset{\text{Ex 4.1}}{\Rightarrow}$ unimodal)

Rmks: For PFS, stronger inequality $\left(\frac{a_k}{\binom{n}{k}}\right)^2 \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}$ holds.

Newton's inequalities.

A distracting riddle: deduce $n=2$ Edrei-Thoma from quadratic formula

"Best" proof of PFS

Give a combinatorial interpretation of the minors

$$\det \begin{pmatrix} a_0 & a_1 & & a_n \\ & \ddots & \ddots & \cdots & \ddots \\ & & \ddots & \ddots & & \ddots \end{pmatrix}_{\underbrace{I}_{\text{rows}},\underbrace{J}_{\text{cols}}}$$

as natural numbers.

Edrei-Thoma matrix

Since det is a signed sum, this usually means canceling out each minus term with a corresponding plus term, and understanding the leftover terms (which will be certain plus terms).

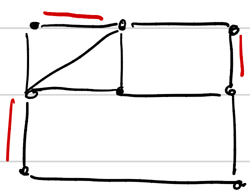
Eg | Lindstrom lemma gives combinatorial interpr'n of determinants.
(as non-intersecting paths $I \to J$)
leftovers
sources sinks

Rem Problem #4 asks you to explore sequences from graph theory
as P.F.S vs log-concave vs unimodal in 3 setups.
strongest weakest.

Def'n | Let G be an edge-weighted undirected graph. Terminology:

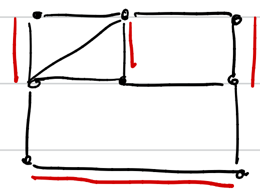
dimer	edge in G
dimer configuration / matching	collection of edges, each vtx used 0 or 1 times
dimer cover / perfect matching	. . . . exactly 1 time.

Eg |



partial matching using 3 dimers

$k = 3$



perfect matching using 4 dimers

$k = 4$

Thm | Heilmann Lieb 1970s

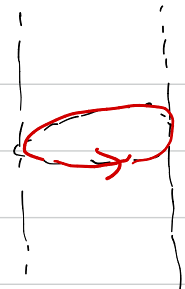
For any G, let

$a_k :=$ # of dimer configs on G using k dimers.

Then (a_k) is a PFS.

Rmk: true $\forall$ G, no planarity etc. assumption.

Setup 1 : infinite cylinder $\ominus$



$\bullet\!-\!\circ$ Bipartite graph $G \subseteq \ominus$
embedded

with choice of positive direction

- For dimer cover π, by convention direct edges $\bullet \to \circ$
  denote by $\pi^\vee$ same edges but directed $\bullet \leftarrow \circ$
- $\forall\, \pi_1, \pi_2$, $\underbrace{\pi_1 \cup \pi_2^\vee}_{\text{double dimer cover}}$ is a union of oriented, simple cycles (and double edges $\bullet \rightleftarrows \circ$)

Defn | $ht(\pi_1, \pi_2) = \left(\#\text{ of positively oriented cycles in } \pi_1 \cup \pi_2^\vee\right) - \left(\#\text{ of neg. oriented cycles}\right)$

"homology class" | total winding #

Typical

$\pi_1 \cup \pi_2^\vee =$ 

$ht = -1$

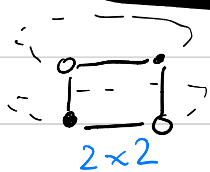
Fact $\forall$ dimer covers π_1, π_2, π_3

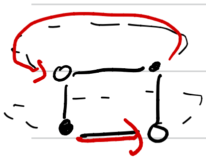
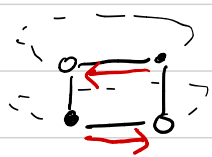
$ht\pi_1 - ht\pi_2 \qquad ht\pi_1 - ht\pi_3 - (ht\pi_2 - ht\pi_3)$

$$ht(\pi_1,\pi_2) = ht(\pi_1,\pi_3) - ht(\pi_2,\pi_3)$$

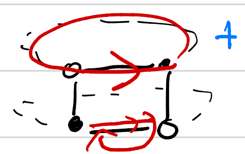
upshot: can treat

$$ht(\pi_1,\pi_2) = ht(\pi_1) - ht(\pi_2)$$

Eg $G =$  admits dimer covers:

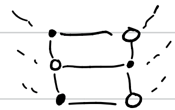
 and 

$\pi_1 \qquad\qquad \pi_2$

$\pi_1 \cup \pi_2^\vee =$ 

$ht(\pi_1) - ht(\pi_2) = +1$

Rev Exercise 4.3

- Explain why $\Pi_1 \cup \Pi_2^\vee$ is always a union of oriented simple cycles.
- for $G =$  and  identify all dimer covers + their relative heights.

Rev Problem 4.1

Let $a_k =$ # of dimer covers of ht k. not just dimer configurations

- Show that (a_k) is a PFS (for any bip cylindrical $G \subseteq \Theta$).

Conjecture of Pylyavskyy

- give combinatorial proofs of unimodality, log-concavity, or PFS

Technical remark: Sps G has edge weights ($\mathbb{R}_{>0}$). Define a_k as a weighted sum. We expect (a_k) is a PFS, and expect that Edrei–Thoma minors are positive sums of monomials in edge wts.

2×2 minors are SO (combinatorially)

Res Problem 4.2 Give combinatorial proof of Heilmann-Lieb thm (what do the minors "count"?), or of other classical real-rootedness theorems. Edrei-Thoma

Setup 2 $G \subseteq \Theta$ is a directed graph embedded on cylinder whose oriented cycles all have positive winding number.

(Galashin - Pylyavskyy Conjecture 6.1) A **k-cycle** on G is a union of k vertex-disjoint simple cycles in G. (cf. cylindrical Lindström lemma)

Res Problem 4.3 Establish that a_k = # of k-cycles in G is a PFS

(give combinatorial proofs)

do edge-weighted case...

If $G \subseteq \mathcal{O}$ is an undirected graph, a cycle-rooted spanning forest

not bip anymore

is a spanning subgraph whose connected components have

# of vertices = # of edges (one more edge than a tree $\Rightarrow$ one cycle)

Setup 3

Kenyon Thm 6.1

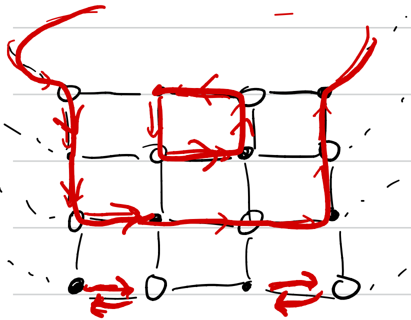
Let c_k = # of CRSF $\subseteq G$ with k essential connected components.

Then $\sum c_k \left(2 - t + \frac{1}{t}\right)^k$ is real-rooted. (essential: cycle has winding #)

Res Prob 4.4

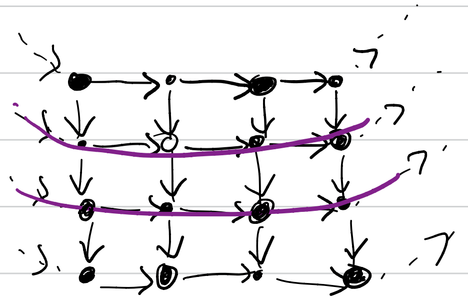
give a combinatorial interpretation of Edrei-Thoma minors in this setting (CRSFs).

bipartite undirected



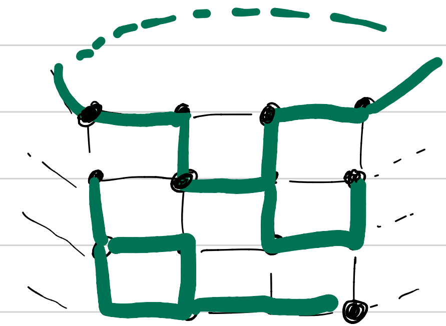
double dimer cover of winding # +1

directed, all cycles ↻



a 2-cycle on the directed graph G (Galashin-Pylyavskyy)

undirected



CRSF with 1 essential component (Kenyon)

Further reading: unimodality surveys by Stanley '89, Brenti '94, Branden 2014

Galashin-Pylyavskyy + setup 2

Kenyon setup 3 + much more

Seymour-Chudnovsky: generalization of Heilmann-Lieb with useful techniques

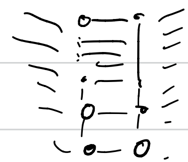
Further exercises (for later)

Rev Ex 4.4

- check real-rootedness for 2×2 and 2×3 grid graphs in setup 1. Bonus: do with edge weights in 2×2 case.

Rev Ex 4.5

- Derive a recurrence describing # of dimer covers of the $2\times n$ cylindrical grid



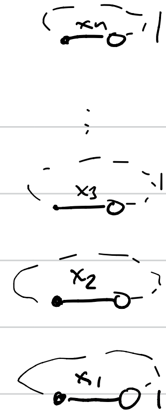
What is the name of this seq of #s?

- Describe a similar recurrence for the edge-weighted $2\times n$ cylindrical grid (pick a good naming convention for the edges)

Rev Ex 4.6

In setup 1,

When G is the graph



(x_i = edge weight)
"loop edges" have weight one

what is a_i, $\sum a_i t^i$?

Rev Ex 4.7

Present a proof of $ht(\pi_1, \pi_3) - ht(\pi_2, \pi_3) = ht(\pi_1, \pi_2)$,
e.g. from Kenyon.

Rev Ex 4.8

Present proof ideas of the Heilmann-Lieb thm
(original source, Seymour-Chudnovsky version, recent updates)