Virtual Resolutions of Monomial Ideals

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August 6, 2020

Introduction

Abstract

In the UMN REU, we explored the relationship between the multi-graded regularity and resolution regularity of virtual resolutions of square-free monomial ideals in $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^2$.

Multigraded Polynomial Rings

Example

The polynomial ring $k[x_1, \ldots, x_n]$ with the "standard grading" is $\mathbb{Z}$-graded, with $\deg(x_i) = 1$. So $\deg(x_1^5 x_2^3) = 8$

Example

Consider the polynomial ring $k[x_0, x_1, y_0, y_1, y_2]$ for $\mathbb{P}^1 \times \mathbb{P}^2$ with $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Then the degrees of the following monomials are

- $\deg(x_0 x_1) = (2, 0)$
- $\deg(x_1^2 y_1 y_2) = (2, 2)$

Minimal Free Resolutions

Let C_0 be an ideal of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^2$.

Definition

A complex $C_0 \xleftarrow{d_0} C_1 \xleftarrow{d_1} C_2 \xleftarrow{d_2} \cdots$ is a minimal free resolution of C_0 if

1. C_i are free modules,
2. It is minimal
3. It is exact

Example of a Resolution

Example

This example is taken from [2]. For I the ideal corresponding to a specific curve in $\mathbb{P}^1 \times \mathbb{P}^2$, we have that the minimal free resolution of I is

$$S^1 \leftarrow \begin{matrix} S(-3,-1)^1 \\ \oplus \\ S(-2,-2)^1 \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-1,-5)^3 \\ \oplus \\ S(0,-8)^1 \end{matrix} \leftarrow \begin{matrix} S(-3,-3)^3 \\ \oplus \\ S(-2,-5)^6 \\ \oplus \\ S(-1,-7)^1 \\ \oplus \\ S(-1,-8)^2 \end{matrix} \leftarrow \begin{matrix} S(-3,-5)^3 \\ \oplus \\ S(-2,-7)^2 \\ \oplus \\ S(-2,-8)^1 \end{matrix} \leftarrow S(-3,-7)^1 \leftarrow 0\,.$$

$$S^1 \leftarrow S^8 \leftarrow S^{12} \leftarrow S^6 \leftarrow S^1 \leftarrow 0.$$

Virtual Resolutions

Definition

[2] A complex $C_0 \xleftarrow{d_0} C_1 \xleftarrow{d_1} C_2 \xleftarrow{d_2} \cdots$ is a virtual resolution if

1. C_i are free modules,
2. $H_i(C_\bullet)$ is irrelevant for $i > 0$.

Remark

For example, for $\mathbb{P}^1 \times \mathbb{P}^2$ the irrelevant ideal is $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1, y_2 \rangle$ But if $f \in B$, then f is zero on the coordinates where x_0 and x_1 are 0 or where b_0, b_1, and b_2 are all zero.

Remark

Over $\mathbb{P}^{\mathbf{n}}$ minimal free resolutions don't accurately reflect the geometry. Virtual free resolutions do.

Example of a Resolution

Example

This example is taken from [2]. For I the ideal corresponding to a specific curve in $\mathbb{P}^1 \times \mathbb{P}^2$, we have that the minimal free resolution of I is

$$S^1 \leftarrow S^8 \leftarrow S^{12} \leftarrow S^6 \leftarrow S^1 \leftarrow 0.$$

However there is a virtual resolution of the form

$$S^1 \leftarrow \begin{matrix} S(-3,-1)^1 \\ \oplus \\ S(-2,-2)^1 \\ \oplus \\ S(-2,-3)^2 \end{matrix} \leftarrow S(-3,-3)^3 \leftarrow 0\,.$$

$$S^1 \leftarrow S^4 \leftarrow S^3 \leftarrow 0.$$

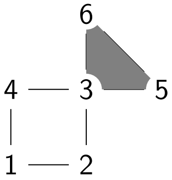
Squarefree Monomial Ideals

Squarefree monomial ideals are a special case of monomial ideas where none of the variables show up in a generator with degree higher than 1.

Definition (Stanley-Reisner Correspondence)

For a simplicial complex Δ on n vertices, define $I_\Delta \subset k[x_1, \ldots, x_n]$ to be the ideal generated by the minimal non-faces.

Example



$$I_\Delta = (x_1x_3, x_1x_5, x_1x_6, x_2x_4, x_2x_5, x_2x_6, x_4x_5, x_4x_6)$$

Saturation

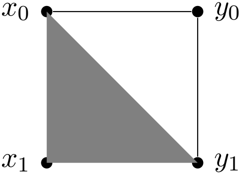
Definition

The <u>saturation</u> of an ideal I by an ideal B is given by

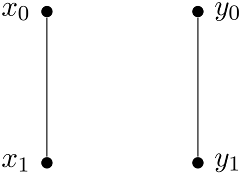
$$I : B^\infty := \left\{ r \in S : r \cdot B^k \subset I \text{ for } k \text{ sufficiently large} \right\}$$

Example

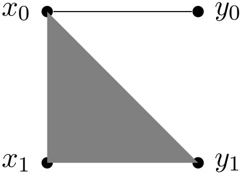
$I =$



$B =$



$I : B^\infty =$



Special Case of Virtual Resolutions

Lemma

The minimal free virtual resolution of I is the minimal free resolution of it's B-saturation.

Proposition

Subvarieties of a product of projective spaces correspond to homogeneous B-saturated radical ideals in the homogeneous coordinate ring

$$\{\textit{Varieties in } \mathbb{P}^{\mathbf{n}}\} \leftrightarrow \{\textit{homogeneous } B\textit{-saturated radical ideals}\}$$

Remark

All monomial ideals are homogeneous and a monomial ideal is radical if and only if it is squarefree.

Multigraded and Resolution Regularity

For a module M, we have a minimal free resolution $M \leftarrow F_1 \leftarrow \cdots$ of M.

Definition ([1])

The mutli-graded regularity $\operatorname{reg}(M)$ of M is an infinite set in $\mathbb{N}^r$.

Definition ([3])

The resolution regularity $\operatorname{res-reg}(M)$ of M is a vector in $\mathbb{N}^r$ given by

$$\operatorname{res-reg}(M)_l = \max \{\mathbf{a}_l : \mathbf{a} + i \cdot e_l \text{ is the degree of a generator in } F_i\}$$

Remark ([4])

The resolution regularity gives a bound on the multigraded regularity. But in general, it does not give the whole multigraded regularity.

Resolution Regularity

$\text{res-reg}(M)_l = \max\{\mathbf{a}_l : \mathbf{a} + i \cdot e_l$ is the degree of a generator in $F_i\}$

Example

$$S^1 \leftarrow \begin{matrix} S(-3,-1)^1 \\ \oplus \\ S(-2,-2)^1 \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-1,-5)^3 \\ \oplus \\ S(0,-8)^1 \end{matrix} \leftarrow \begin{matrix} S(-3,-3)^3 \\ \oplus \\ S(-2,-5)^6 \\ \oplus \\ S(-1,-7)^1 \\ \oplus \\ S(-1,-8)^2 \end{matrix} \leftarrow \begin{matrix} S(-3,-5)^3 \\ \oplus \\ S(-2,-7)^2 \\ \oplus \\ S(-2,-8)^1 \end{matrix} \leftarrow S(-3,-7)^1 \leftarrow 0\,.$$

$$\text{res-reg}(S/I) = (2,7)$$

A Problem to Consider

Question

How is $\operatorname{reg}(S/(I : B^\infty))$ related to $\operatorname{res-reg}(S/I)$?

Calculating Resolution Regularity in M2

Macaulay2 has a package for multigraded regularity. We made code for resolution regularity.

```
    resRegularityHelper = (r,l) -> (
    max for k in keys betti r list (
        k#1#l - k#0
        )
    )

resRegularity = (r) -> (
    d := degreeLength ring r;
    for l from 0 to (d-1) list (
        resRegularityHelper(r,l)
        )
    )
```

Enumerating $\mathbb{P}^1 \times \mathbb{P}^1$

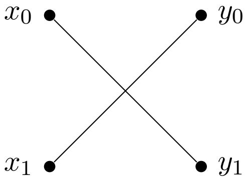
Δ_I	$\overline{\Delta_I \setminus \Delta_B}$	$\operatorname{reg} I : B^\infty$	res-reg I
{12}	{∅}	{{0,0}}	{0,0}
{13}	{13}	{{0,0}}	{0,0}
{12,13}	{13}	{{0,0}}	{0,0}
{13,14}	{13,14}	{{0,1}}	{0,1}
{13,34}	{13}	{{0,0}}	{0,0}
{12,34}	{23}	{{0,0}}	{0,0}
{13,24}	{13,24}	{{0,1},{1,0}}	{1,1}
{12,13,14}	{13,14}	{{0,1}}	{0,1}
{12,13,23}	{13,23}	{{1,0}}	{1,0}
{12,13,24}	{13,24}	{{0,1},{1,0}}	{0,1}
{12,13,34}	{13}	{{0,0}}	{0,0}
{12,13,14,23}	{13,14,23}	{1, 1}	{1, 1}
{12,13,24,34}	{13,24}	{{0,1},{1,0}}	{0,0}
{12,13,23,34}	{13,23}	{{1,0}}	{1,0}
{13, 14, 23, 24}	{13, 14, 23, 24}	{{1,1}}	{1,1}

Enumerating $\mathbb{P}^1 \times \mathbb{P}^1$

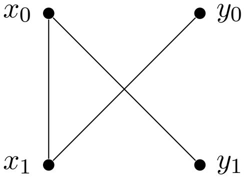
Δ_I	$\overline{\Delta_I \setminus \Delta_B}$	$\operatorname{reg} I : B^\infty$	res-reg I
{13,23,24}	{13,23,24}	{{1,1}}	{1,1}
{12,13,14,23,24}	{13,14,23,24}	{{1,1}}	{1,1}
{12,13,14,23,34}	{13,14,23}	{{1,1}}	{1,1}
{12,13,14,23,24,34}	{13,14,23,24}	{{1,1}}	{1,1}
{123}	{123}	{{0,0}}	{0,0}
{123,14}	{123,14}	{{0,1}}	{0,1}
{123,34}	{123}	{{0,0}}	{0,0}
{123,14,34}	{123, 14}	{{0,1}}	{0,1}
{123,14,24}	{123, 14, 24}	{{1,1}}	{1,1}
{123,124}	{123, 124}	{{0,1}}	{0,1}
{123,124,34}	{123, 124}	{{0,1}}	{0,1}
{123,134}	{123, 134}	{{0,0}}	{0,0}
{123,134,24}	{123, 134, 24}	{{0,1}}	{0,1}
{123,124,134}	{123, 124, 134}	{{0,0}}	{0,1}
{123,124,134,234}	{123,124,134,234}	{{1,1}}	{1,1}

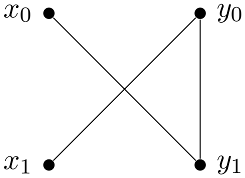
Example in $\mathbb{P}^1 \times \mathbb{P}^1$

$$\text{reg}(I) = \mathbb{N}^2 \setminus \{(0,0)\}$$



The following resolution regularities are $(0,1)$ and $(1,0)$.





Random $\mathbb{P}^1 \times \mathbb{P}^2$ in M2

We made code to test random examples for ideals in $\mathbb{P}^1 \times \mathbb{P}^2$.

```
X = toricProjectiveSpace(1)**toricProjectiveSpace(2)
R = ring X
P=newRing(R,DegreeRank=>1)
phi=map(R,P)
L={...}--degrees of minimial generators of ideal.
I = randomSquareFreeMonomialIdeal(L,P)
print resolutionInformation phi(I);
```

Further Directions

One might use a combinatorial interpretation of local cohomology to give a cominatorial interpretation of multigraded regularity. There already exists one for resolution regularity [5].

Proposition [6]

Let $\Sigma \subset \Delta$ be simplicial complexes, and let $\mathbf{a} \in \mathbb{Z}$, $F_+ = \mathsf{supp}_+(\mathbf{a})$ and $F_- = \mathsf{supp\_}(\mathbf{a})$ Then

$$H^i_J(k[\Delta]] \cong \tilde{H}^{i-1}(||\mathsf{star}_\Delta(F_+) - ||\Sigma|, ||\mathsf{del}_{\mathsf{star}_\Delta(F_+)}(F_-))|| - ||\Sigma|)$$

where $||\Delta||$ denotes the geometric realization of Δ.

Acknowledgements

This research was conducted 2020 REU UMN supported by NSF RTG grant DMS-1745638. We thank Jay Yang and Elizabeth Kelley for their support.

References

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Conclusion

Questions?