q-Analogues of Rational Numbers

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Outline

The q-Integers

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Definition

For each $n \in \mathbb{N}$, define the polynomial $[n]_q \in \mathbb{Z}[q]$:

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Remark: Substituting $q = 1$ gives n.

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- **Order:** Define a partial order on rational functions by $\frac{a(q)}{b(q)} > \frac{c(q)}{d(q)}$ if $a(q)d(q) - b(q)c(q)$ has all positive coefficients.
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  If $\frac{a}{b} > \frac{c}{d}$, we might expect $\left[\frac{a}{b}\right]_q > \left[\frac{c}{d}\right]_q$.
- **Convergence:** If $\frac{a_n}{b_n} \to \lambda \in \mathbb{R}$ irrational, we might expect $\left[\frac{a_n}{b_n}\right]_q$ to "converge" in some sense, and moreover be independent of the sequence.

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Exercise 9.1: Find an example where this definition does not satisfy the order property. That is, find two fractions $\frac{a}{b} > \frac{c}{d}$ where $[a]_q[d]_q - [b]_q[c]_q$ has some negative coefficient.

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A *continued fraction* is an expression consisting of nested fractions, like this:

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Remark: These are not unique. For example, $\frac{7}{4}$ is also equal to $[1, 1, 2, 1]$. Requiring an even number of coefficients makes it unique.

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If $\frac{r}{s} = [a_1, a_2, \dots, a_{2n}]$, then define

$$\left[\frac{r}{s}\right]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{\cdots + \cfrac{q^{a_{2n-1}}}{[a_{2n}]_{q^{-1}}}}}$$

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Fact: The only time this agrees with the "naive guess" is for $\left[\frac{n+1}{n}\right]_q = \frac{[n+1]_q}{[n]_q}$.

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Theorem

This definition of $\left[\frac{a}{b}\right]_q$ does satisfy the order and convergence properties.

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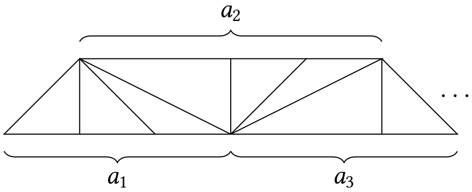

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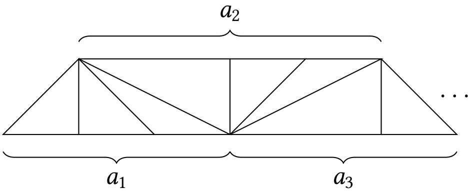

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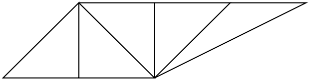

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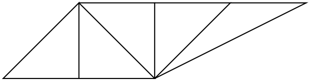

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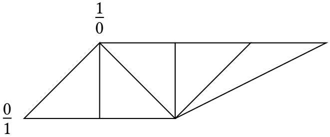

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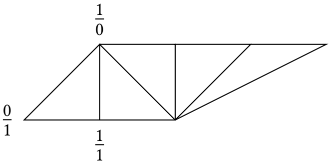

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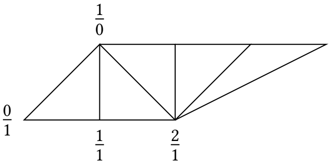

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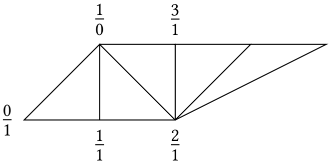

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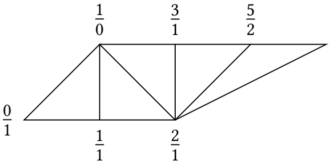

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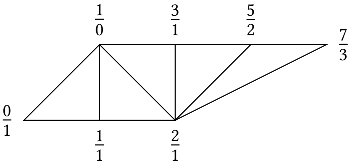

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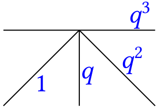

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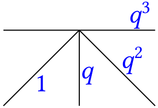

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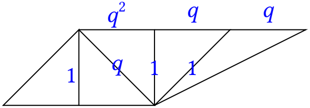

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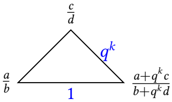

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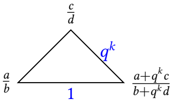

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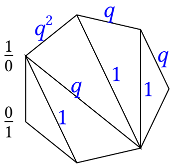

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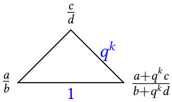

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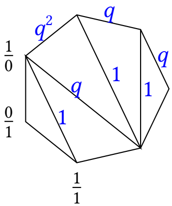

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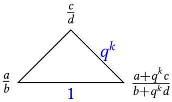

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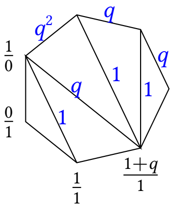

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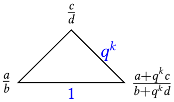

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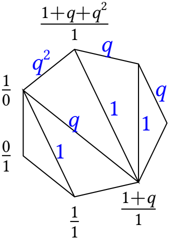

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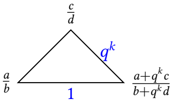

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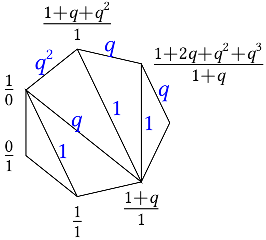

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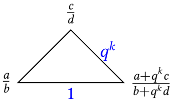

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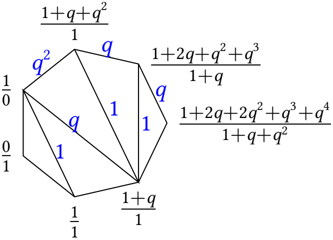

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(*b*) Prove that $[1, 1, \cdots, 1]$ is always a ratio of Fibonacci numbers.

(*c*) Use the triangulation method to compute $\left[\frac{5}{3}\right]_q$ and $\left[\frac{8}{5}\right]_q$.

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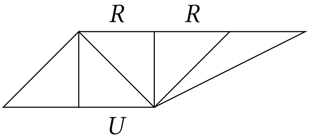

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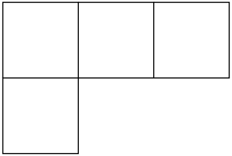
$G_{7/3} =$ 

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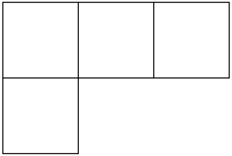

Caution: In the literature, "snake graph" refers to a slightly different, but related, construction. The construction above is called the "dual snake graph" corresponding to a triangulation.

Lattice Paths

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If G is a snake graph, let $L(G)$ be the set of all paths in G from the south-west corner to the north-east corner using only right and up steps.

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Theorem [Schiffler, Çanakçi]

$$\left|L(G_{r/s})\right| = r \quad \text{and} \quad \left|L(\widehat{G}_{r/s})\right| = s$$

The notation $\widehat{G}_{r/s}$ means the snake graph from the continued fraction $[a_2, a_3, \ldots, a_n]$.

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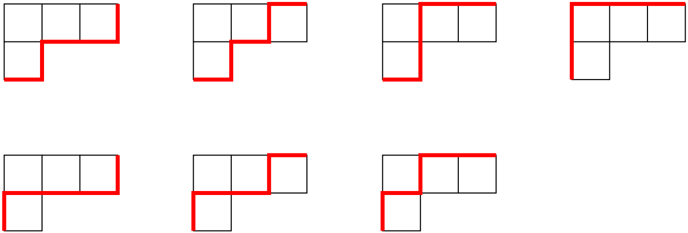
Example: The 7 lattice paths in $G_{7/3}$ are



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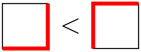

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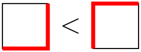
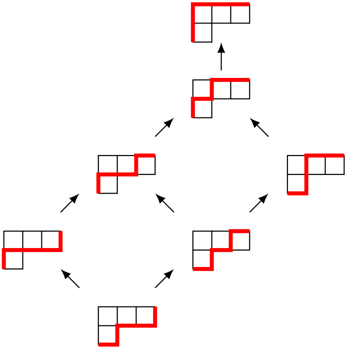


Example: $L(G_{7/3})$



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Theorem

Let $\left[\frac{r}{s}\right]_q = \frac{R(q)}{S(q)}$. Then:

1. The coefficient of q^k in $R(q)$ is the number of lattice paths in $G_{r/s}$ of height k.
2. The coefficient of q^k in $S(q)$ is the number of lattice paths in $\widehat{G}_{r/s}$ of height k.

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REU Exercise 9.3: Write down all the lattice paths in $G_{8/5}$ and draw the Hasse diagram.

(It should agree with Exercise 9.2(d)!)

What Else Do They Count?

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Some Suggested Presentations:

- T-paths:
  Schiffler, R. "A cluster expansion formula (A_n case)". *The Electronic Journal of Combinatorics* 15.1 (2008) R64

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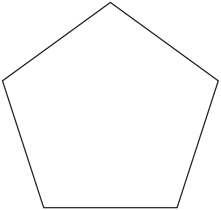
- All of the above:
  Claussen, A., "Expansion Posets for Polygon Cluster Algebras". *arxiv:2005.02083* (2020)

Outline

The Cluster Algebra of a Polygon

The Cluster Algebra of a Polygon

- Consider an n-gon



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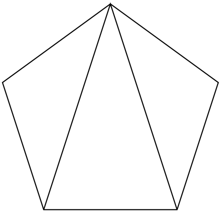

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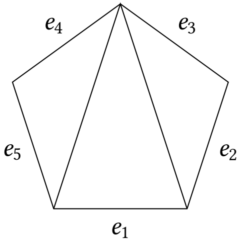
- Consider an n-gon
- Choose a triangulation
- Label the edges $e_1, \ldots, e_n$



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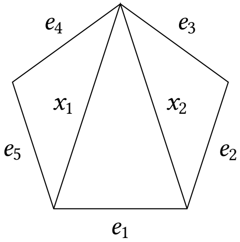

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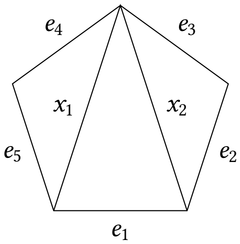

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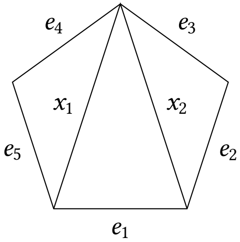

- The cluster algebra is a subring of $\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_{n-3}, e_1, \ldots, e_n)$.

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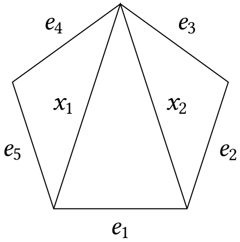

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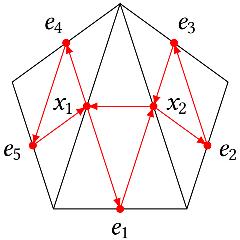

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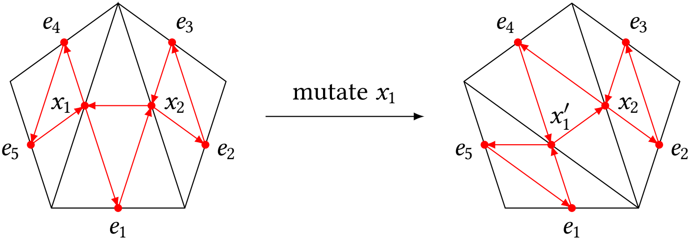
- The quiver is ...

Mutations are “Flips”

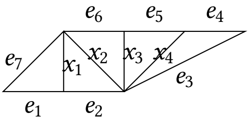


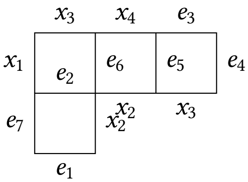
$$x_1' = \frac{e_1 e_4 + e_5 x_2}{x_1}$$

Snake Graphs

Snake Graphs

Construct the snake graph from a triangulation as we described before.





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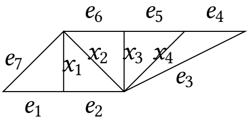

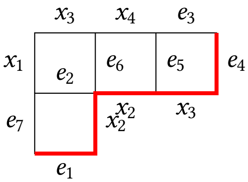

$\mathrm{wt}(p) = e_1 e_4 x_2^2 x_3$

Each lattice path p corresponds to a monomial, called the *weight* of the path, denoted $\mathrm{wt}(p)$.

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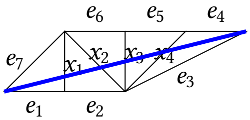

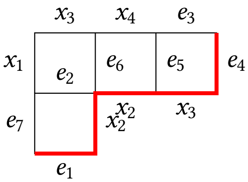

Each lattice path p corresponds to a monomial, called the *weight* of the path, denoted $\mathrm{wt}(p)$.

Theorem

The cluster variable of the "longest edge" (crossing all diagonals) is

$$\frac{1}{x_1 x_2 \cdots x_n} \sum_{p} \mathrm{wt}(p)$$

Term Count

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Corollary

The cluster variable of the longest edge in $T_{r/s}$ has exactly r terms.

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REU Exercise 8.3: Consider the cluster variable represented by the blue arc on the previous slide.

(a) Compute the Laurent polynomial expression for this cluster variable using the formula in the theorem on the previous slide.

(b) Compute the same expression using a sequence of mutations (you should get the same answer!).

The F-Polynomial

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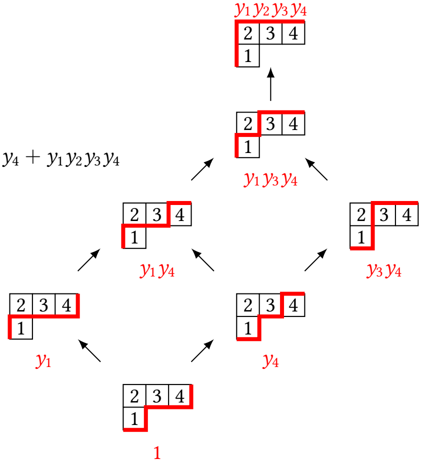
Label the faces of the snake graph by $y_1, \ldots, y_n$, and label the lattice paths by monomials in the y's:

$$F = 1 + y_1 + y_4 + y_1 y_4 + y_3 y_4 + y_1 y_3 y_4 + y_1 y_2 y_3 y_4$$



Relation with q-Rationals

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(b) The coefficient of q^k in the numerator of $\left[\frac{r}{s}\right]_q$ counts the number of terms of degree k in the F-polynomial of the corresponding cluster variable.

Some REU Problems

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REU Problem 9.0: (Tie-in with Gregg's Problem) Is there a combinatorial description of the L's from Gregg's talk related to the q-rationals?

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REU Problem 9.1: ("Unimodality") It is conjectured that the numerators (and denominators) of the q-rationals are *unimodal.* Any progress towards proving this would be nice, even for some non-trivial class of specific examples.

In light of Chris' talk, you could also try to prove log-concavity.

Outline

Infinite Continued Fractions

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There is a notion of infinite continued fractions. For an infinite sequence $a_1, a_2, a_3, \ldots$, define a sequence of rational numbers (called *convergents*):

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Examples:

- $\sqrt{2} = [1, 2, 2, 2, \ldots]$
- $\sqrt{3} = [1, 1, 2, 1, 2, 1, 2, \ldots]$
- $\varphi = \frac{1}{2}(1 + \sqrt{5}) = [1, 1, 1, 1, \ldots]$

Convergence Property

The first few convergents of $\sqrt{2} = [1, 2, 2, 2, \ldots]$ are

$$\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots$$

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The q-versions are

$$\left[\frac{3}{2}\right]_q = 1 + q^2 - q^3 + q^4 - q^5 + \cdots$$

$$\left[\frac{7}{5}\right]_q = 1 + q^3 - 2q^5 + q^6 + 3q^7 + \cdots$$

$$\left[\frac{17}{12}\right]_q = 1 + q^3 - 2q^5 + 2q^6 - q^8 + \cdots$$

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The coefficients eventually “stabilize”. The terms in blue remain the same in all later terms in the sequence.

Another REU Problem

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REU Problem 9.3: (Very open-ended) Almost nothing is known about the coefficients of these power series for "q-real numbers", except for a few select specific examples computed in the original paper. Is there a pattern to these coefficients that can be predicted? Is there a combinatorial interpretation? Is it related to cluster algebras and snake graphs (see below)?

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More Further Reading:

- Section 7 of the paper "Cluster Algebras and Continued Fractions" (mentioned earlier)

Extra Reading

Here are some more papers that could be used for a presentation in weeks 3 and 4:

- Morier-Genoud, S., Ovsienko, V., "q-Deformed Rationals and q-Continued Fractions". *Forum of Mathematics, Sigma* Vol. 8 (2020)
- Morier-Genoud, S., Ovsienko, V., "On q-Deformed Real Numbers". *Experimental Mathematics* (2019): 1-9