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Fix n. $t_{ij} := (i\,j) \in S_n$ Specify a sequence of transpositions

$t_{i_1 j_1}\ t_{i_2 j_2} \cdots t_{i_m j_m}$

$\uparrow \quad \uparrow \quad \uparrow$

probability $p_1, p_2, \ldots, p_m$ that we include this transposition

in a subsequence $S \subset [m]$

$$S \longmapsto \sigma_S \in S_n$$

$$\mathbb{P}(S) = \prod_{i \in S} p_i \prod_{i \notin S} (1-p_i)$$

GOAL: Specify $\{(t_{i_k j_k}, p_k)_{k=1}^m\}$

such that σ has uniform distribution on S_n.

$$\underbrace{\underbrace{\underset{\frac{n-1}{n}}{t_{n,n-1}}\ \underset{\frac{n-2}{n-1}}{t_{n-1,n-2}} \cdots \underset{\frac{1}{2}}{t_{12}}}_{\text{puts a uniformly random element in position 1}}\ \underbrace{\underset{\frac{n-2}{n-1}}{t_{n-1,n}}\ t_{n-1,n-2} \cdots \underset{1/2}{t_{23}}}_{\text{puts a unif. random element in pos 2}} \cdots}_{\text{achieves unif. dist. in } \binom{n}{2} \text{ steps}}$$

Q: Can we do this in less than $\binom{n}{2}$ steps?

NB: Need at least $n \log n$ steps, just to create every $\sigma \in S_n$.

~~But it doesn't~~ appear to help to have non-adjacent transpositions!

FACT: Need at least $n-1$ of the p_i to be $1/2$?

proof: Let $M = (m_{ij})$ where $m_{ij} = \mathbb{P}(\sigma_i = j)$

so need $\underset{\text{of rank 1}}{M} = \begin{bmatrix} 1/n & \cdots & 1/n \\ & \vdots & \\ 1/n & \cdots & 1/n \end{bmatrix}$. We start with $\underset{\text{of rank } n}{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}$, and

But $t_{ij}(p)$ acts as $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \begin{bmatrix} p_{ij} & 1-p_{ij} \\ 1-p_{ij} & p_{ij} \end{bmatrix} & \\ & & & 1 \end{pmatrix}$ which can only lower rank when $p_{ij} = 1/2$ $\blacksquare$