P. Pylyavskyy CPS, Nov. 7, 2014

Schur functions

λ partition

SSYT $T = \begin{array}{ccccc} 1 & \le & 1 & \le & 2 \\ \wedge & & \wedge & & \\ 3 & \le & 3 & \le & 4 \\ \wedge & & & & \\ 4 & & & & \end{array}$ $\qquad x^T = x_1^2 x_2 x_3^2 x_4^2$

Schur function $s_\lambda := \sum_{\substack{\text{SSYT } T \\ \text{of shape } \lambda}} x^T$

Represent Schubert classes in cohomology of Grassmannians

In K-theory, one has Grothendieck polynomials

and dual Grothendieck polynomials

Set-valued tableaux

$\begin{array}{ccccc} 1,2 & \le & 2 & \le & 2,4 \\ \wedge & & \wedge & & \\ 3,4 & \le & 5 & & \end{array}$ $\qquad x^T = x_1 x_2^3 x_3 x_4^2 x_5$

Groth. poly

$$G_\lambda := \sum_{\substack{\text{set-valued } T \\ \text{of shape } \lambda}} x^T = s_\lambda + \underbrace{\cdots}_{\text{higher degree terms}}$$

Reverse plane partitions

$\begin{array}{ccccccc} 1 & \le & 1 & \le & 2 & \le & 2 \\ \wedge\!| & & & & & & \\ 1 & \le & 2 & \le & 2 & \le & 4 \\ \wedge\!| & & & & & & \\ 2 & \le & 3 & & & & \\ \wedge\!| & & & & & & \\ 4 & & & & & & \end{array}$ $\qquad x^T := x_1^2 x_2^4 x_3^1 x_4^2 = \prod_i x_i^{\#\text{columns containing } i}$

($\neq x^T$ from before!)

dual Groth. poly.

$$g_\lambda := \sum_{\substack{\text{rev. plane} \\ \text{partitions } T \\ \text{of shape } \lambda}} x^T$$

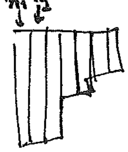
Jacobi–Trudi: $e_n = s_{(1,1,\ldots,1)}$ $\qquad e_1 = x_1 + x_2 + \ldots$

$e_2 = x_1x_2 + x_1x_3 + \ldots$

$\vdots$

THM: $s_\lambda = \det(e_{\lambda'_i - i + j})$

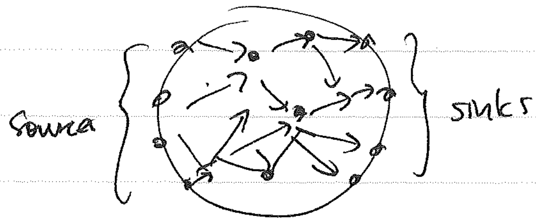
$\lambda'_1 \; \lambda'_2$



e.g. $s_{(3,2)} = \det \begin{bmatrix} e_2 & e_3 & e_4 \\ e_1 & e_2 & e_3 \\ \underset{0}{\overset{\shortparallel}{e_{-1}}} & \underset{1}{\overset{\shortparallel}{e_0}} & e_1 \end{bmatrix}$

proof of Jacobi-Trudi:

Lindström's Lemma Given a digraph which is acyclic, planar, edge-weighted, with k sources and k sinks identified on one side / other side

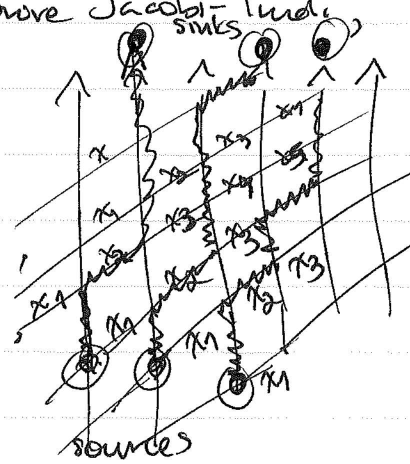


$$\text{weight}(\underset{p}{\text{path}}) := \prod_{\substack{\text{edges}\\ \text{in } p}} \text{weight}(\text{edge})$$

Let $a_{ij} := \displaystyle\sum_{\substack{\text{paths}\\ p \text{ from } i \text{ to } j}} \text{weight}(p)$

Then $\det(a_{ij}) = \displaystyle\sum_{\substack{\text{non}\cancel{\text{crossing}}\text{intersecting}\\ k\text{-tuples of paths}\\ P_{ii}}} \text{weight}(P)$

To prove Jacobi-Trudi,



$\leftrightarrow$

2	2	2
4	5	

For the dual Grothendiecks, $g_\lambda \overset{\text{THM}}{=} \det\left(\binom{\lambda_i'-1}{\lambda_i'-1} e_{\lambda_i'-i+j} + \dots + \binom{\lambda_i'-1}{0} e_{1-i+j} \right)$

e.g. $g_{(3,2)} = \begin{vmatrix} e_2+e_1 & e_3+e_2 & e_4+e_3 \\ e_1+1 & e_2+e_1 & e_3+e_2 \\ 0 & 1 & e_1 \end{vmatrix}$ $g_{(2,2,1)} = \begin{vmatrix} e_3+2e_2+e_1 & e_4+2e_3+e_2 \\ e_1+1 & e_2+e_1 \end{vmatrix}$

PROBLEM: Find a Lindstrom lemma style proof.