What is Algebraic Combinatorics?
A general counting problem
Four properties
An algebraic approach
Summary

Algebraic Combinatorics
Using algebra to help one count

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Max and Rose Lorie Lecture Series
George Mason University
January 29, 2010
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Outline

1. What is Algebraic Combinatorics?
2. A general counting problem
3. Four properties
4. An algebraic approach
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What is Algebraic Combinatorics?

- **Combinatorics** is the study of finite or discrete objects, and their structure.
- Counting them is **enumerative combinatorics**.
- One part of **algebraic combinatorics** is using **algebra** to help you do enumerative combinatorics.
What is Algebraic Combinatorics?

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What is Algebraic Combinatorics?

- Combinatorics is the study of finite or discrete objects, and their structure.
- Counting them is enumerative combinatorics.
- One part of algebraic combinatorics is using algebra to help you do enumerative combinatorics.
Example: enumerating subsets up to symmetry

We’ll explore an interesting family of examples:

Enumerating subsets, up to symmetry.

This has many interesting properties,

- some easier,
- some harder (without algebra!).
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Enumerating **subsets**, up to **symmetry**.

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A group permuting the first $n$ numbers

Let $[n] := \{1, 2, \ldots, n\}$, permuted by the **symmetric group** $\mathfrak{S}_n$ on $n$ letters.

Let $G$ be any subgroup of $\mathfrak{S}_n$, thought of as some chosen symmetries.
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EXAMPLE: $G=$ cyclic symmetry, with $n = 6$
Counting $G$-orbits of subsets

Let’s count the set

$$2^{[n]} := \{ \text{all subsets of } [n] \}$$

or equivalently,

**black-white colorings** of $[n]$,

but only **up to equivalence** by elements of $G$.

I.e. let’s count the $G$-orbits

$$2^{[n]} / G$$
Counting $G$-orbits of subsets

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EXAMPLE: black-white necklaces

For $G$ the cyclic group of rotations as above, $G$-orbits of colorings of $[n]$ are sometimes called necklaces.
All the black-white necklaces for $n = 6$

In this case, $|2^n / G| = 14$. 
More refined counting of $G$-orbits

Let’s even be more refined: count the sets

\[ ([n]_k) := \{ \text{all k-element subsets of } [n] \} \]

or equivalently,

black-white colorings of $[n]$ with $k$ blacks,

but again only up to equivalence by elements of $G$.

I.e. we want to understand

\[ c_k := |( [n]_k ) / G | \]

= number of $G$-orbits of black-white colorings of $[n]$ with $k$ blacks.
More refined counting of $G$-orbits

Let’s even be more refined: count the sets

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$$c_k := |\binom{[n]}{k} / G|$$

number of $G$-orbits of black-white colorings of $[n]$ with $k$ blacks.
The refined necklace count for $n = 6$

Here $(c_0, c_1, c_2, c_3, c_4, c_5, c_6) = (1, 1, 3, 4, 3, 1, 1)$. 
QUESTION: What can we say in general about the sequence $c_0, c_1, c_2, \ldots, c_n$?

AN ANSWER: They share many properties with the case where $G$ is the trivial group, where the $c_k$ are the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}.$$
The basic question

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$$
\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}
$$
The binomial coefficients

Recall what binomial coefficient sequences look like:

\[
\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}
\]

| n = 0 | 1 |
| n = 1 | 1 1 |
| n = 2 | 1 2 1 |
| n = 3 | 1 3 3 1 |
| n = 4 | 1 4 6 4 1 |
| n = 5 | 1 5 10 10 5 1 |
| n = 6 | 1 6 15 20 15 6 1 |
PROPERTY 1 (the easy one)

SYMMETRY: For any permutation group $G$, one has $c_k = c_{n-k}$

This follows from
- complementing the subsets, or
- swapping the colors in the black-white colorings.
PROPERTY 1 (the easy one)

SYMmetry: For any permutation group $G$, one has $c_k = c_{n-k}$

This follows from
- complementing the subsets, or
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PROPERTY 2 (the hardest one)

UNIMODALITY: (Stanley 1982)

\[ c_0 \leq c_1 \leq \ldots \leq c_{\frac{n}{2}} \geq \ldots \geq c_{n-1} \geq c_n \]

e.g.

\[ 1 \leq 1 \leq 3 \leq 4 \geq 3 \geq 1 \geq 1 \]

Nontrivial, but fairly easy with some algebra. Currently only known in general via various algebraic means.
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PROPERTY 3 (not so hard, but a bit surprising)

ALTERNATING SUM: (de Bruijn 1959)

\[ c_0 - c_1 + c_2 - c_3 + \cdots \text{ counts self-complementary } G\text{-orbits.} \]

e.g. there are \( 1 - 1 + 3 - 4 + 3 - 1 + 1 = 2 \)

self-complementary black-white necklaces for \( n = 6 \):
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Wait! How was that like binomial coefficients?

It’s easy to see that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = (1 + (-1))^n = 0$$

and there are no self-complementary subsets $S$ of $[n]$. 
PROPERTY 4 (not so hard, but also a bit surprising)

GENERATING FUNCTION: (Redfield 1927, Polya 1937)

\[ c_0 + c_1 q + c_2 q^2 + c_3 q^3 + \cdots + c_n q^n \]

is the average over all \( g \) in \( G \) of the very simple products

\[ \prod_{\text{cycles } C \text{ of } g} (1 + q^{\lvert C \rvert}) \]
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\[ \mathbf{G} = \{\begin{array}{ccccccc}
\end{array}\} \]

\[
(1 + q^1)^6 = q^0 + 6q^1 + 15q^2 + 20q^3 + 15q^4 + 6q^5 + q^6
\]

\[
(1 + q^6)^1 = q^0 + q^6
\]

\[
(1 + q^3)^2 = q^0 + 3q^2 + 3q^4 + q^6
\]

\[
(1 + q^2)^3 = q^0 + 2q^3 + q^6
\]

\[
(1 + q^3)^2 = q^0 + 2q^3 + q^6
\]

\[
(1 + q^6)^1 = q^0 + q^6
\]

\[
\begin{array}{ccccccc}
6q^0 & +6q^1 & +18q^2 & +24q^3 & +18q^4 & +6q^5 & +6q^6
\end{array}
\]

\[
\times \frac{1}{6}
\]

\[
1q^0 + 1q^1 + 3q^2 + 4q^3 + 3q^4 + q^5 + 1q^6
\]
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Summary
In the algebraic approach, instead of thinking of numbers like $|2^{[n]}/G|$ and $c_k = |\binom{[n]}{k}/G|$ as **cardinalities** of sets, one tries to re-interpret them as **dimensions** of vector spaces.

Hopefully these vector spaces are natural enough that one can prove

- **equalities** of cardinalities via vector space **isomorphisms**,
- **inequalities** via vector space **injections** or **surjections**,  
- **identities** via **trace identities**, etc.
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Linearize!

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In the algebraic approach, instead of thinking of numbers like $|2^{[n]}/G|$ and $c_k = |([n]/G)|$ as **cardinalities** of **sets**, one tries to re-interpret them as **dimensions** of **vector spaces**.

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Let $V = \mathbb{C}^2$ have a $\mathbb{C}$-basis

$$\{ w, b \}$$

white black

Then

$$V^\otimes n := \underbrace{V \otimes \cdots \otimes V}_{n \text{ tensor positions}}$$

has its tensor positions labelled by $[n]$, and has a $\mathbb{C}$-basis $\{ e_S \}$ indexed by

- black-white colorings of $[n]$, or
- subsets $S$ of $[n]$. 
Tensor products and colorings

Let $V = \mathbb{C}^2$ have a $\mathbb{C}$-basis

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Tensor products and colorings

Let \( V = \mathbb{C}^2 \) have a \( \mathbb{C} \)-basis

\[
\{ \begin{array}{c}
\| \quad \| \\
\text{white} & \text{black}
\end{array}
\}
\]

Then

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V \otimes^n := \underbrace{V \otimes \cdots \otimes V}_{n \text{ tensor positions}}
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has its tensor positions labelled by \([n]\), and has a \( \mathbb{C} \)-basis \( \{e_S\} \) indexed by

- black-white colorings of \([n]\), or
- subsets \( S \) of \([n]\).
A typical basis tensor $e_S$

E.g. For $n = 6$ and the subset $S = \{1, 4, 5\}$, one has the basis element of $V^\otimes 6$

$$e_{\{1, 4, 5\}} = b \otimes w \otimes w \otimes b \otimes b \otimes w$$

or for short, just

$$e_{\{1, 4, 5\}} = bwwbbw$$
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Quick tensor product reminder

Recall tensor products are **multilinear**, that is, linear in each tensor factor.

E.g. for any constants $c_1, c_2$ in $\mathbb{C}$ one has

$$b \otimes w \otimes (c_1 \cdot w + c_2 \cdot b) \otimes b \otimes b \otimes w$$

$$= c_1 \cdot (b \otimes w \otimes w \otimes b \otimes b \otimes w)$$

$$+ c_2 \cdot (b \otimes w \otimes b \otimes b \otimes b \otimes w)$$
Recall tensor products are *multilinear*, that is, linear in each tensor factor.

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\[ b \otimes w \otimes (c_1 \cdot w + c_2 \cdot b) \otimes b \otimes b \otimes w = c_1 \cdot (b \otimes w \otimes w \otimes b \otimes b \otimes w) + c_2 \cdot (b \otimes w \otimes b \otimes b \otimes b \otimes w) \]
The subspace of $G$-invariants

The subgroup $G$ of $\mathfrak{S}_n$ acts on $V \otimes \underbrace{n}_{\text{tensor}}$ by **permuting the tensor positions**.

Consider the subspace of $G$-invariants

$$(V \otimes \underbrace{n}_{\text{tensor}})^G.$$

This has a $\mathbb{C}$-basis naturally indexed by

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Example

E.g. for $n = 6$ with $G = \text{cyclic rotations}$, the element

$$wwbwwb + bwwbww + wbwwbw \in (V \otimes 6)^G$$

corresponds to the necklace shown:

CONCLUSION: $|2^n/G| = \dim_{\mathbb{C}} (V \otimes n)^G$
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CONCLUSION: $|2^n/G| = \dim_{\mathbb{C}} \left( V \otimes^n \right)^G$
Better yet, if one defines subspaces

\[ V_k \otimes n := \mathbb{C}\text{-span of } \{e_S \text{ with } |S| = k \} \]

then

- one has a direct sum decomposition \( V \otimes n = \bigoplus_{k=0}^{n} V_k \otimes n \),
- the group \( G \) acts on each \( V_k \otimes n \), and
- \( c_k := |( [n] ) / G| = \dim_{\mathbb{C}} ( V_k \otimes n )^G \).
Interpreting the $c_k$’s

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$$V_k^\otimes n := \mathbb{C}\text{-span of } \{e_S \text{ with } |S| = k\}$$

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Interpreting the $c_k$'s

Better yet, if one defines subspaces

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Algebraic Combinatorics
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- \( c_k := |\binom{[n]}{k}/G| = \dim_{\mathbb{C}} (V_k^n)^G \).
This gives a good framework for understanding the $c_k$. We’ve naturally **linearized** this picture:
Silly proof of Property 1: SYMMETRY

We want to show

\[ c_k = c_{n-k} \]

Or equivalently,

\[ \dim_\mathbb{C} (V_k^n)^G = \dim_\mathbb{C} (V_{n-k}^n)^G. \]

So we’d like a \( \mathbb{C} \)-linear isomorphism

\[ (V_k^n)^G \rightarrow (V_{n-k}^n)^G. \]
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Any $\mathbb{C}$-linear map

$$t : V \rightarrow V$$

gives rise to a $\mathbb{C}$-linear map

$$t : V \otimes^n \rightarrow V \otimes^n$$

acting \textit{diagonally}, i.e. the same in each tensor position.
Schur-Weyl duality

Such maps commute with the $G$-action permuting the tensor positions.

$$v_1 \otimes v_2 \otimes v_3 \xrightarrow{t} t(v_1) \otimes t(v_2) \otimes t(v_3)$$

$$\downarrow g = (12) \quad \downarrow g = (12)$$

$$v_2 \otimes v_1 \otimes v_3 \xrightarrow{t} t(v_2) \otimes t(v_1) \otimes t(v_3)$$
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Silly proof of SYMMETRY (cont’d)

Let $t : V \to V$ swap the basis elements $\{w, b\}$, so on tensors it also swaps them, e.g.

$$t(bwbbwb) = wbwwbw.$$ 

Note that $t^2 = 1$, so $t$ gives a $\mathbb{C}$-linear isomorphism

$$V^\otimes_k \to V^\otimes_{n-k}$$

which restricts to a $\mathbb{C}$-linear isomorphism

$$(V_k^\otimes)^G \to (V_{n-k}^\otimes)^G,$$

as desired to show $c_k = c_{n-k}$. QED
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We want to show that

\[ c_0 - c_1 + c_2 - c_3 + \cdots \]

counts **self-complementary** \(G\)-orbits.

Begin with this observation:

**PROPOSITION:** The number of **self-complementary** \(G\)-orbits is the **trace** of the color-swapping map \(t\) from before, when it acts on \((V \otimes n)^G\).
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**PROPOSITION:** The number of **self-complementary** \(G\)-orbits is the \textit{trace} of the color-swapping map \(t\) from before, when it acts on \((V \otimes n)^G\).
Proof.

- $t$ permutes the basis of $(V^\otimes n)^G$ indexed by $G$-orbits of black-white colorings, and
- $t$ fixes such a basis element if and only if this $G$-orbit is self-complementary. QED

For example, with $n = 6$ and $G = \text{cyclic rotation}$, $t$ fixes this basis element of $(V^\otimes 6)^G$

$$wbwbwb + bwwwwb + bbwwwb + bbbwww + wbbbbw + wwwbbw$$

as it is a sum over the $t$-stable $G$-orbit shown below:
Not-so-silly proof (cont’d)

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as it is a sum over the $t$-stable $G$-orbit shown below:
What does this have to do with $c_0 - c_1 + c_2 - \cdots$?

Well, inside $GL(V)$,

$$t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- are both diagonalizable and have eigenvalues $+1, -1$,
- so they must be **conjugate** within $GL(V)$,
- so $t, s$ must act on $V^\otimes n$ and on $(V^\otimes n)^G$ by $\mathbb{C}$-linear maps which are conjugate.
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so $t, s$ must act on $V^\otimes n$ and on $(V^\otimes n)^G$ by $\mathbb{C}$-linear maps which are conjugate.
What does this have to do with $c_0 - c_1 + c_2 - \cdots$? Well, inside $GL(V)$,

$$t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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Recall that $\text{Tr}(AB) = \text{Tr}(BA)$ implies conjugate transformations have the same trace:

$$\text{Tr}(PAP^{-1}) = \text{Tr}(P^{-1} \cdot PA) = \text{Tr}(A).$$

Thus $s$, $t$ must act with the same trace on $(V^\otimes n)^G$.

We know from the previous Proposition that this trace for $t$ is the number of self-complementary $G$-orbits.

So it suffices to apply the following fact with $q = -1$...
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PROPOSITION: For any eigenvalue $q$ in $\mathbb{C}$, the element $s(q) = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$ acts on $(V^\otimes n)^G$ with trace

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In particular, for $q = -1$, the element $s = s(-1)$ acts with trace

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- \(s(q)\) fixes \(w\).
- \(s(q)\) scales \(b\) by \(q\).
- Hence \(s(q)\) scales any \(e_S\) in which \(|S| = k\) by \(q^k\), e.g.

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s(q)(bwbbbw) = qb \otimes w \otimes qb \otimes qb \otimes qb \otimes w = q^4 \cdot bwbbbw.
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- Hence \(s(q)\) scales all of \(V_k \otimes^n\) by \(q^k\),
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- So \(s(q)\) acts on \((V \otimes^n)^G = \bigoplus_k (V_k \otimes^n)^G\) with trace \(\sum_k c_k q^k\).

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Not-so-silly proof (cont’d)

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A proof of Property 2: UNIMODALITY

We want to show that for $k < \frac{n}{2}$, one has

$$c_k \leq c_{k+1}$$

So we’d like a $\mathbb{C}$-linear injective map

$$(V_k \otimes^n)^G \hookrightarrow (V_{k+1} \otimes^n)^G.$$ 

Maybe we should look for an injective map

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The only natural injection

There is only one **obvious candidate** for such an injection $U_k : V_k^n \hookrightarrow V_{k+1}^n$, namely define

$$U_k(e_S) := \sum_{T \supset S : \|T\| = k+1} e_T$$

E.g. for $n = 6, k = 2$, one has

$$U_2(bwbwww) = bbwww + bwbwwb + bwbwbw + bwbwwb$$

Easy to check $U_k$ commutes with $\mathfrak{S}_n$ permuting positions. But why is $U_k$ **injective**?
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A cute injectivity argument

There are several arguments for this, but here’s a cute one.

PROPOSITION: For $k < \frac{n}{2}$, the operator $U_k^t U_k$ on $V_k^{\otimes n}$ turns out to be **positive definite**, i.e. all its (real) eigenvalues are **strictly** positive.

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Quick review of positive (semi-)definiteness

Recall that a real symmetric matrix $A = A^t$

- always has only real eigenvalues,
- is positive semidefinite if they’re all nonnegative, or equivalently, $x^t Ax \geq 0$ for all vectors $x$,
- is positive definite if they’re all positive, or equivalently, if $x^t Ax > 0$ for all nonzero vectors $x$,
- is always positive semidefinite when $A = B^t B$ for some rectangular matrix $B$, since

$$x^t Ax = x^t B^t Bx = |Bx|^2 \geq 0$$
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A cute injectivity argument (cont’d)

**PROOF** that $U_k^t U_k$ is positive definite.

- Check (on each $e_S$) that

  $$U_k^t U_k - U_{k-1} U_{k-1}^t = (n - 2k) \cdot I_{V_k \otimes n}$$

- Hence

  $$U_k^t U_k = U_{k-1} U_{k-1}^t + (n - 2k) \cdot I_{V_k \otimes n}$$

- First term $U_{k-1} U_{k-1}^t$ is positive *semidefinite*.
- Second term $(n - 2k) \cdot I_{V_k \otimes n}$ is positive *definite* as $k < \frac{n}{2}$.
- Hence the sum $U_k^t U_k$ is positive *definite*. QED
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A proof of Property 4: GENERATING FUNCTION

(To be flipped through at lightning speed during the talk; read it later, if you want!)

We want to show

$$\sum_{k=0}^{n} c_k q^k = \frac{1}{|G|} \sum_{g \in G} \left( \prod_{\text{cycles } C \text{ of } g} (1 + q^{|C|}) \right)$$

Such averages over the group are ubiquitous due to the following easily-checked fact.
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Such averages over the group are ubiquitous due to the following easily-checked fact.
An idempotent projector

PROPOSITION: When a finite group $G$ acts linearly on a vector space $W$ over a field in which $|G|$ is invertible (nonzero), the map $W \ni w \mapsto \frac{1}{|G|} \sum_{g \in G} g(w)$ is

- **idempotent**, i.e. $\pi^2 = \pi$, and
- $\pi$ **projects** onto the subspace of $G$-invariants $W^G$. 
One then has a second ubiquitous and easily-checked fact.

PROPOSITION: In characteristic zero, the trace $Tr(\pi)$ of an idempotent projector onto a linear subspace is the dimension of that subspace.
Trace of idempotent = dimension of image

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PROPOSITION: In characteristic zero, the trace $\text{Tr}(\pi)$ of an idempotent projector onto a linear subspace is the dimension of that subspace.
Putting two idempotent facts together

Apply these two facts to the idempotent projector $\pi = \frac{1}{|G|} \sum_{g \in G} g$ onto the $G$-fixed subspace of each $W = V_k^\otimes n$:

$$\sum_k c_k q^k = \sum_k \dim_{\mathbb{C}} (V_k^\otimes n)^G q^k = \sum_k \text{Tr} \left( \pi|_{V_k^\otimes n} \right) q^k = \frac{1}{|G|} \sum_{g \in G} \left( \sum_k \text{Tr}(g|_{V_k^\otimes n}) q^k \right).$$

It only remains to show

$$\sum_k \text{Tr}(g|_{V_k^\otimes n}) q^k = \prod_{\text{cycles } C \text{ of } g} (1 + q^{|C|}).$$
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Trace of $g$ counts colorings monochromatic on its cycles

To see

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\sum_k \text{Tr}(g|_{V_k^\otimes n}) q^k = \prod_{\text{cycles } C \text{ of } g} (1 + q^{|C|})
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note that

- any $g$ in $G$ permutes the basis for $V_k^\otimes n$ indexed by black-white colorings,
- and $g$ fixes such a coloring if and only if it is monochromatic on each cycle $C$ of $g$. 
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To see

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Proof by example

E.g. $g = (12)(34)(567)$ in $\mathcal{S}_7$ fixes these colorings/tensors:

<table>
<thead>
<tr>
<th></th>
<th>12</th>
<th>34</th>
<th>567</th>
</tr>
</thead>
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<td>$ww$</td>
<td>$www$</td>
<td>$+q^2$</td>
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<tr>
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<td>$ww$</td>
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<td>$ww$</td>
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<td>$+q^2 \cdot q^3$</td>
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QED
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E.g. \( g = (12)(34)(567) \) in \( S_7 \) fixes these colorings/tensors:

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