Hodge theory and matroids

Ref: Adiprasito, Huh & Katz "Hodge theory for combinatorics"

1. Unimodality, log-concavity
2. Rota-Heron-Welsh conj.
3. Mason conj.
4. Huh, Katz, & Bucalossi, & a better conj.
5. Truncation reduction
6. Getting into the Hodge story

1. Given \((a_0, a_1, \ldots, a_r) \in \{1, 2, 3, \ldots\}\)

say they are unimodal if \(a_0 \leq a_1 \leq \ldots \leq a_k \geq \ldots \geq a_r\) for some \(k\)

log-concave if \(a_i^2 \geq a_{i-1} a_{i+1}\) for \(i = 1, 3, \ldots, r-1\)

\[
\left( \Rightarrow \frac{a_i}{a_{i-1}} > \frac{a_{i+1}}{a_i} \Rightarrow \text{unimodal} \right)
\]

2. For a graph, \(G\), the chromatic polynomial

\[
\chi_G(t) := \# \{\text{proper vertex-colorings}\}
\]

with \(t\) colors

\[
= t(t-1)(t-2) \quad \text{in this case}
\]

\[
= t^4 - 4t^3 + 5t^2 - 2t
\]

is always a polynomial in \(t\)

CONJ (Read, Haggard) Its coefficients are unimodal/log-concave

\(\text{e.g. (1,4,5,2) here}\)

CONJ (Rota-Heron, Welsh) Same holds more generally for characteristic polynomial of any matroid \(M\)

\[
\chi_M(t) := \sum_{\text{flats } F \text{ of } M} \mu(F, F) t^{r(M)-r(F)}
\]
(2) \( (V, E) \)
Recall a graph \( G \)

\[ M = \begin{bmatrix}
1 & +1 & -1 & 0 & +1 & 0 \\
2 & -1 & 0 & +1 & 0 & 0 \\
3 & 0 & +1 & -1 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & -1
\end{bmatrix} \]

\[ \text{in a vector space over a field} \ b \]

Matroids \( M \) have their lattice of flats \( F \)

\[ := \text{subsets} \ F \text{ indexing vectors closed under lin. span} \]

\[ \mu(\phi, F) \text{ circled} \]

Axioms:
- \( \phi, \epsilon F \subset E \)
- \( I \cap X, J \subset X \Rightarrow I \cap X \subset J \subset X \)
- \( I, J \subset X, \ |I| > |J| \Rightarrow \exists \, \ell \in I \setminus J \)

\[ X_M(t) = 1 + 5t + 6t^2 + \sum_{\text{flats } F} \mu(\phi, F) t^{r(M) - r(F)} \]

In general, graph \( G \) with matroid \( M \) has \( X_G(t) = \sum_{\text{comps of } G} X_M(t) \)

**Easy Möbius inversion argument**

3. **Mason (1972)** If \( a_k = \# \text{indep. sets of size } k \) in matroid \( M \), then \( (a_0, a_1, \ldots, a_r) \) is log-concave

\[ e.g. \text{above} \ (1, 4, 6, 3) \]

\[ \begin{bmatrix} 1 & 2 & 4 & 3 \end{bmatrix} \]

4. In fact, there is a better conjecture/theorem...

A matroid \( M \) has \( X_M(t) = (t-1)^{-1} X_M(t) \)

\[ \text{reduced characteristic polynomial} \]

\[ := \sum_{\text{flats } F \text{ of } M} \mu(\phi, F) t^{r(M) - r(F)} \]

for any choice of \( e \in F \)
e.g. \( M \) above, choosing \( e = a \)

\[
\begin{array}{ccc}
5b + c & +2 \\
5b & -3t \\
\theta & t^2 \\
\end{array}
\]

choosing \( e = d \)

\[
\begin{array}{ccc}
abc +2 \\
\theta & -3t \\
\phi & t^2 \\
\end{array}
\]

\[
\chi_M(t) = t^3 - 4t^2 + st - 2 = \left( t-1 \right) \left( t^2 - 3t + 2 \right)
\]

PROPP (Bylawski, Lenz 1977-2013) There is a matroid construction \( M \mapsto M \times e := (M^* \cup \{e\})^* \)

such that \( \chi_{M \times e}(t) \) has signless coefficient sequence \( (a_0, a_1, \ldots, a_n) \)

where \( a_k = \# \text{indep sets in } M \text{ of size } k \)

\[
\begin{array}{c|c|c|c}
\text{M} & \text{M}^* & \text{M}^* \cup \{e\} \\
\hline
\begin{array}{ccc}
b & c & \theta \\
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{array} & \begin{array}{ccc}
a^* & b^* & c^* \\
+1 & +1 & -1 \\
\end{array} & \begin{array}{ccc}
a^* & b^* & c^* & e^* \\
+1 & +1 & -1 & +1 \\
\end{array} \\
\end{array}
\]

\[
\chi_{M \times e}(t) = t^3 - 4t^2 + 6t - 3
\]

\[
= (t-1) (t^2 - 3t + 3)
\]

\[
\chi_M(t) = \frac{\chi_{M \times e}(t)}{\chi_M(t)}
\]

\[
\text{Proof of PROP is easy using } \chi_M(t), \sum_{\text{indep sets } I} t^{|I|}
\]

both being Tutte polynomial evaluations

\[
T_M(x,y) = \sum_{\text{indep sets } I} t^{|I|}
\]

\[
x=3 \quad y=0
\]

\[
x=1+4t \quad y=1
\]
Things started happening a few years ago...

**THM (Huh 2012)** For matroids $M$ representable by vectors in characteristic zero, $\overline{X}_M(t)$ has log-concave signless coeff. seq.

$$\Rightarrow \text{coeffs of } (1+t)(a_0+a_1t+\ldots+a_{r-1}t^{r-1}) \text{ log-concave}$$

**THM (Huh, Katz 2012)** Same for matroids $M$ repd over any field.

**THM (Lenz 2013)** Therefore Mason's conj. holds for $M$ repd over any field.

**THM (Adiprasito, Huh, Katz 2015)** Every matroid $M$ has $\overline{X}_M(t)$ with log-concave signless coeff. seq.

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How did they do it? Somehow mimicking a calculation in a Chasing hyperplane for the wonderful compactification of the arrangement complement $k^n - U^{ht}$, when $M$ is realized over $k$...

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5. **The truncation reduction:**

One only needs to show $a_{r-1} \leq a_{r-2}$ at the end of the signless coeff. seq. $(a_0, a_1, \ldots, a_{r-3}, a_{r-2}, a_{r-1})$ for $\overline{X}_M(t)$, because the matroid operation $M \mapsto \text{tr}(M)$ truncation of $M$ to 1 lower rank (generic projection of vectors) has signless coeff. seq. $(a_0, a_1, \ldots, a_{r-3}, a_{r-2})$ for $\overline{X}_{\text{tr}(M)}(t)$.

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E.g., $M = \begin{pmatrix} a & b & c & d \\ +1 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(5) OUTLINE for more of the Hodge story...

6. An amazing thing
7. Its Hodge properties
8. How they help

Recall a matroid $M \mapsto$ lattice of flats $\mapsto$ reduced characteristic polynomial

\[ \chi_M(t) = a_0 + a_1 t + \cdots + a_r t^r \]

Signless coefficients $(a_0, a_1, \ldots, a_r)$

\[ \chi_M(t) = t^3 - 5t^2 + 8t - 4 = (t-1)(t^2 - 4t + 4) \]

\[ \Rightarrow (a_0, a_1, a_2) = (1, 4, 4) \]

**Thm (A-H-K 2015)** $(a_0, a_1, \ldots, a_{r-2})$ is log-concave: $a_k^2 \geq a_{k-1} a_{k+1}$

Truncation reduction means we only need to show

\[ a_{r-2}^2 \geq a_{r-3} a_{r-1} \]

6. $A(M):= \text{"Chow ring of M" (Feichtner & Yuzvinsky 2004) cohomology of wonderful configuration...}$

\[ A(M) = \left[ \mathbb{Z}[X_F] \right]_{\text{proper flat of } M} / \left( \mathbb{Z} X_{F_1} G_1 : F \subset G, G \not\subset F \right) \]

Stanley-Reisner ring for proper part of lattice of flats

Prop: $A(M) = \bigoplus_{i=0}^{r-1} A_i(M) = A_0(M) \oplus A_1(M) \oplus \cdots \oplus A_{r-1}(M)$ with

\[ \deg(X_F ; X_{F_2} \cdots X_{F_{r-1}}) = 1 \]

\[ \forall \text{ maximal flags } \quad F_1 \subset F_2 \subset \cdots \subset F_{r-1} \]

Prop: The elements $\alpha_i := \alpha_i$ for any $i \in F$

\[ \beta_i := \sum_{F_i \subset F} X_F - \alpha_i \]

in $A(M)$ satisfy

\[ a_i = \deg(\beta_i \alpha_{r-1-i}) \]
Example: $M = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ above has

$$A(M) = Z[x_a, x_b, x_c, x_d, x_e, x_{abc}, x_{ade}, x_{bd}, x_{be}, x_{cd}, x_{ce}]$$

$$\begin{array}{c}
A_0 \\
A_1 \\
A_2
\end{array}$$

Can check, e.g.

$$x_a x_{ade} = x_e x_{ade}$$

using $x_{abc} (x_a - x_e) = 0$

$$x_{ade} (x_a + x_{abc} - x_e = x_{be} - x_{ce})$$

$$x_a x_{ade} - x_e x_{ade}$$

Can check $\deg(x_{bd}) = -1$

$$\begin{array}{c}
\deg(x_a^2) = -1 \\
\deg(x_a x_{bc}) = -1 \\
\deg(x_{abc}^2) = -1 \\
\deg(x_{b}^3) = -2
\end{array}$$

Then $\alpha = \alpha_a = x_a + x_{abc} + x_{ade}$

$$\beta = x_b + x_c + x_d + x_e + x_{bd} + x_{be} + x_{cd} + x_{ce}$$

have $\alpha^2 = x_a^2 + x_{abc}^2 + x_{ade} + 2 (x_a x_{abc} + x_a x_{ade} + x_{abc} x_{ade})$

$$\begin{array}{c}
\deg(-1-1-1+2(1+1)) = 0
\end{array}$$

$$\alpha \beta = (x_a + x_{abc} + x_{ade}) (x_b + x_c + x_d + x_e + x_{bd} + x_{be} + x_{cd} + x_{ce})$$

$$\begin{array}{c}
\deg(1+1+1+1) = 4
\end{array}$$

$$\beta^2 = x_b^2 + x_c^2 + x_d^2 + x_e^2 + x_{bd}^2 + x_{be}^2 + x_{cd}^2 + x_{ce}^2 + 2 (x_b x_{bd} + x_b x_{be} + x_c x_{cd} + x_e x_{ce} + x_d x_{bd} + x_d x_{cd} + x_e x_{be} + x_e x_{ce})$$

$$\deg(-2+(-2)+(-2)+(-2)+1+1+1+1+2(8)) = 4$$

$$= \alpha_e$$
7. The Hodge properties: (the majority of the work in A-H-K 2015):

- **THM (Poincaré duality over \( \mathbb{Z} \))** The bilinear pairing
  \[
  A^k(M) \times A^{n-k}(M) \rightarrow \mathbb{Z}
  \]
  \[
  (x, y) \mapsto \deg(x \cup y)
  \]
  is non-degenerate, i.e., it induces an isomorphism \( A^{n-k}(M) \cong \text{Hom}(A^k(M), \mathbb{Z}) \).

- **THM (Hard Lefschetz property)** When one extends scalars in \( A(M) \) to \( \mathbb{R} \), there are elements in \( A^i(M) \) with a property called ampleness that makes this map an isomorphism for \( k = \frac{n}{2} \):
  \[
  A^k(M) \overset{\sim}{\longrightarrow} A^{n-k}(M)
  \]
  \[
  x \longmapsto \int_M x \cdot \alpha^{n-2k}
  \]

- **THM (Hodge-Riemann relations)** For ample \( \alpha \in A^i(M) \), when one restricts the quadratic form on \( A^k(M) \) defined by
  \[
  Q(x) = \deg(x \cdot \alpha^{n-2k})
  \]
  to the primitive part of \( A^k(M) \), it becomes positive definite if \( k \) is even, negative definite if \( k \) is odd.

8. How does this help?

We want

\[
\begin{align*}
(a_{t-3}, a_{t-2}, a_{t-1}) \\
\deg(\alpha^{t-3}) & \deg(\alpha^{t-2}) & \deg(\beta^2)
\end{align*}
\]

to satisfy \( a_{t-2} \geq a_{t-3}, a_{t-1} \)

**Prop:** \( \beta \in A^i(M) \) is not ample, but is a limit \( \lim_{t \to 0} \beta_t = \beta \) with \( \beta_t \) ample (\( \beta \) is nef)

Now we can show (*) holds with strict inequality for \( \alpha, \beta_t \) using the Hodge-Riemann relations for the ample \( \beta_t \in A^i(M) \):

Since \( \alpha \) is not a multiple of \( \beta_t \) in \( A^i(M) \), the 2-plane \( \{ \alpha + \lambda \beta_t \} \) inside \( A^i(M) \) has the quadratic form

\[
Q(x) = \deg(x \cdot \beta_t^{n-3} \cdot x) \text{ restricted to it indefinite}
\]

(it contains the line \( \lambda \beta_t \) where it is positive definite, plus some primitive part of \( A^i(M) \) where it is negative).
Consequently if we write down the Gram matrix for $Q$ with respect to the choice of basis $\alpha, \beta$ of this plane

$$
\begin{bmatrix}
\alpha & \beta \\
\alpha^t & Q(\alpha, \alpha) = \text{deg}(\alpha^2 \beta_{12}^{-3}) \\
\beta^t & Q(\beta, \alpha) = \text{deg}(\alpha^2 \beta_{13}^{-2})
\end{bmatrix}
$$

Then it should have negative determinant, that is

$$\text{deg}(\alpha^2 \beta_{12}^{-3})^2 \geq \text{deg}(\alpha^2 \beta_{13}^{-2}) \cdot \text{deg}(\beta_{13}^{-1})$$

as desired.