39

OUTLINE :

- Catalan numbers,
  W-Catalan,
  and W-q-Catalan
- The $GL_n(\mathbb{F}_q)$-analogue
- Hurwitz factorizations in $\mathfrak{S}_n$
  and the Goulden-Jackson cactus formula
- A $GL_n(\mathbb{F}_q)$-analogue

DEFINITION: We saw that the Catalan number

$$\mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n}$$

generalizes to the

q-Catalan number $\frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q$,

but also for W a finite real reflection group in $GL_n(\mathbb{R})$ acting irreducibly on $V=\mathbb{R}^n$,

define the W-Catalan number

$$\boxed{\mathrm{Cat}(W) := \prod_{i=1}^{n} \frac{h+d_i}{d_i}}$$

where $S=\mathbb{C}[x_1,\ldots,x_n]$ has

$S^W=\mathbb{C}[f_1,\ldots,f_n]$ polynomial

with degrees $d_1 \leq \ldots\ldots \leq d_n =: h$

and the W-q-Catalan number

$$\boxed{\mathrm{Cat}(W,q) := \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q}} \qquad \left(\begin{array}{c} \overset{?}{\in} \mathbb{Z}[q] \\ \overset{?}{\in} \mathbb{N}[q] \\ \ldots \text{ meaning??} \end{array} \right)$$

41

EXAMPLE:

Type A_{n-1} : To make $\mathfrak{S}_n$ acting on $\mathbb{R}^n$ permuting coordinates act irreducibly, consider the action on $V = \{x \in \mathbb{R}^n : x_1 + \dots + x_n = 0\}$
$\cong \mathbb{R}^{n-1}$

Then $S = \mathbb{C}[x_1, \dots, x_n] / (x_1 + \dots + x_n)$

has $S^{\mathfrak{S}_n} = \mathbb{C}[e_2, e_3, \dots, e_n]$
degrees $2, 3, \dots, n =: h$

and $$\mathrm{Cat}(A_{n-1}) = \frac{(n+2)(n+3)\cdots(n+n)}{2 \cdot 3 \cdots n} = \frac{1}{n+1}\binom{2n}{n}$$
$$= \mathrm{Cat}(n)$$

$$\mathrm{Cat}(A_{n-1}, q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

= the q-Catalan from before

Cat(W) counts something ...

THM (D. Bessis 2003) For irreducible real reflection groups W, with Coxeter system $(W, \underset{\{s_1,\dots,s_n\}}{\overset{}{\underset{\shortparallel}{S}}})$

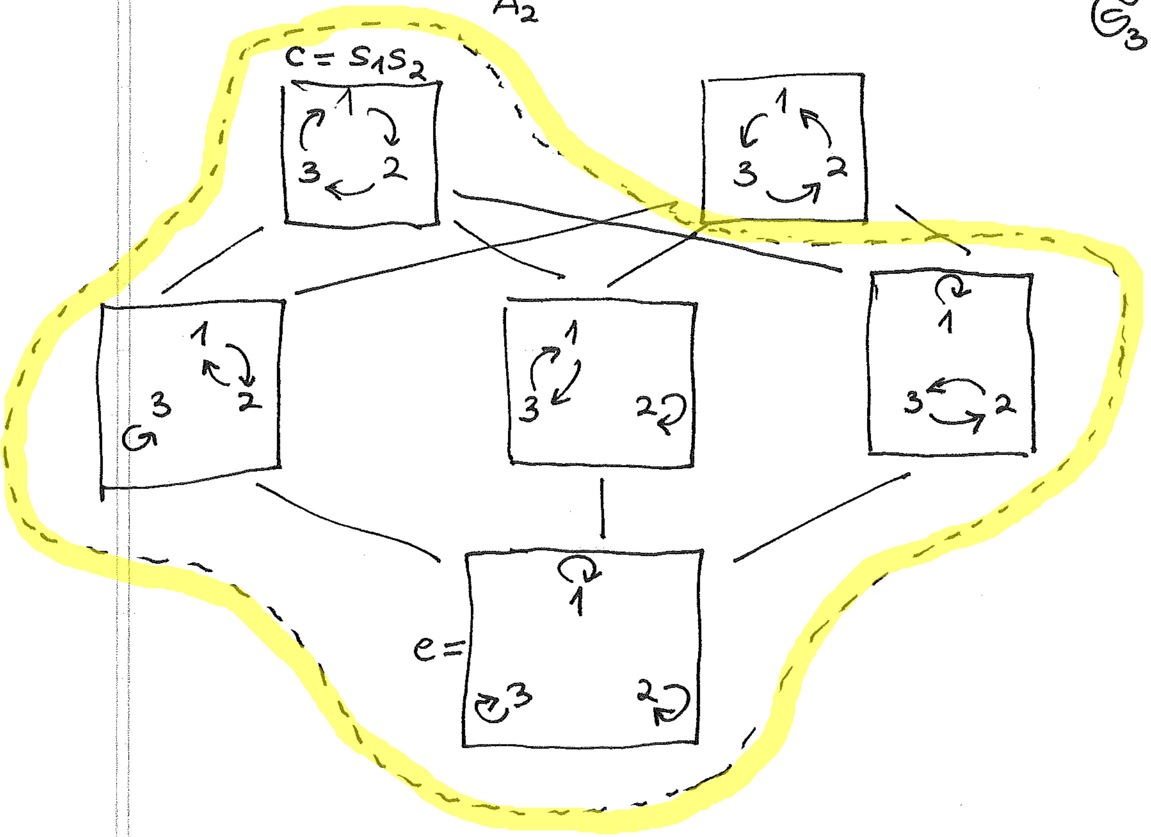
$$\mathrm{Cat}(W) = |NC(W)|$$

where $NC(W) := \{W\text{-"noncrossing partitions"}\}$

$:= \{ w \in W$ lying on a shortest path between the identity e and a Coxeter element $c = s_1 s_2 \cdots s_n$ in the Cayley graph for W using all reflections as generators $\}$

EXAMPLE: $\mathrm{Cat}(\underset{\underset{A_2}{\shortparallel}}{\mathfrak{S}_3}) = \frac{1}{4}\binom{2\cdot 3}{3} = 5$

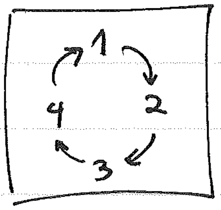
$(\underset{\underset{\mathfrak{S}_3}{\shortparallel}}{W}, \underset{\underset{\{\underset{\underset{(1,2)}{\shortparallel}}{s_1}, \underset{\underset{(2,3)}{\shortparallel}}{s_2}\}}{\shortparallel}}{S})$



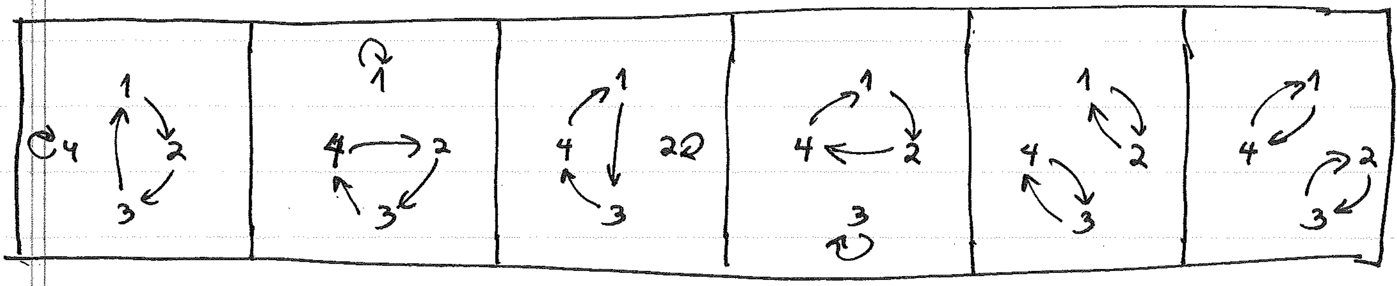
EXAMPLE: $\mathrm{Cat}(\underset{\underset{A_3}{\shortparallel}}{\mathfrak{S}_4}) = \frac{1}{5}\binom{8}{4} = 14 = |NC(\mathfrak{S}_4)|$

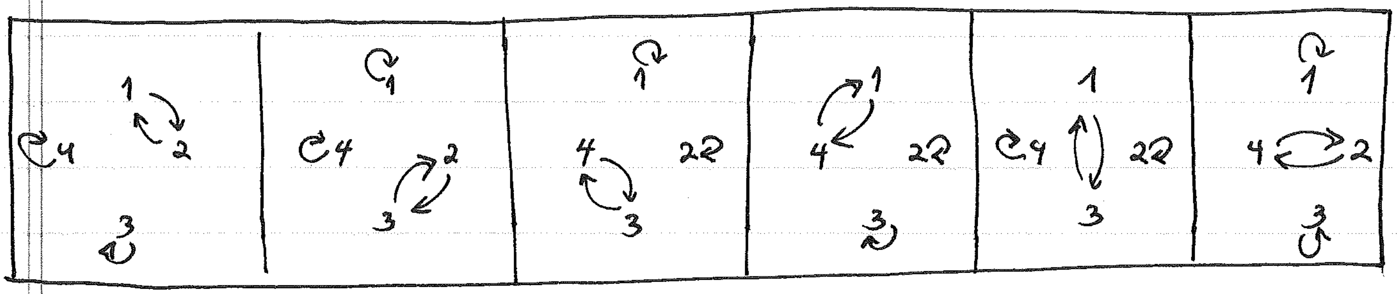
$$(W, S) = (\mathfrak{S}_4, \underset{\underset{\{\underset{(1,2)}{\underset{\shortparallel}{s_1}},\ \underset{(2,3)}{\underset{\shortparallel}{s_2}},\ \underset{(3,4)}{\underset{\shortparallel}{s_3}}\}}{\shortparallel}}{S})$$

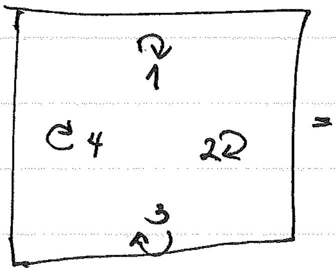
noncrossing set partitions of $\{1, 2, \ldots, \underset{\underset{4}{\shortparallel}}{n}\}$



$= c = s_1 s_2 s_3$







$= e$

$\mathrm{Cat}(W,q)$ gives a CSP ...

THM (Bessis-R.)
This triple $(X, X(q), C)$

$$X := NC(W)$$

$$X(q) := \mathrm{Cat}(W,q)$$

$$C := \langle c \rangle \cong \mathbb{Z}/h\mathbb{Z} \quad \text{conjugating the elements of } NC(W) \quad w \xmapsto{d} c^d w c^{-d}$$

exhibits the CSP.

EXAMPLE: $W = \mathfrak{S}_4$, $h=4$ $\quad \zeta = e^{\frac{2\pi i}{4}} = i$

$$\mathrm{Cat}(\mathfrak{S}_4,q) = \frac{1}{[5]_q}\begin{bmatrix}8\\4\end{bmatrix}_q = 1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}$$

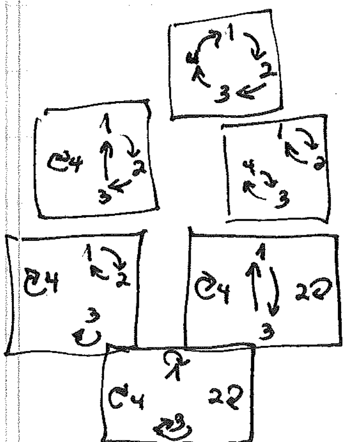
$$\equiv 6+2q^1+4q^2+2q^3 \mod q^4-1$$

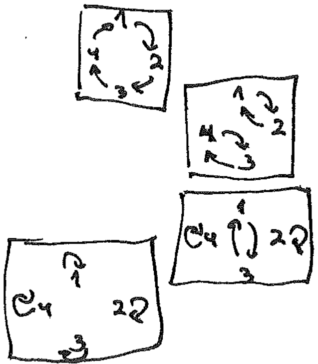
$q=\zeta^0=1$ $\rightsquigarrow$ $14 = |X^{c^0}|$

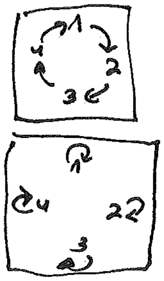
$q=\zeta^2=-1$ $\rightsquigarrow$ $6-2+4-2=6 = |X^{c^2}|$

$q=\zeta^1=i$ $\rightsquigarrow$ $6+2i-4-2i=2 = |X^{c^1}|$

C-
6 orbits







So why is $\underset{\displaystyle\prod_{i=1}^{n}\frac{[h+d_i]_q}{[d_i]_q}}{\underset{\shortparallel}{\mathrm{Cat}(W,q)}}$ $\in \mathbb{Z}[q]$? $\in \mathbb{N}[q]$?

Invariant theory again !

THEOREM (I. Gordon, Berest-Etingof-Ginzburg 2003)

For every irreducible real reflection group W acting on $S=\mathbb{C}[x_1,\dots,x_n]$

there exist $\theta_1,\dots,\theta_n \in S$

- each homogeneous of degree $h+1$,
- giving a system of parameters for S, i.e. $S/(\underline{\theta}) := S/(\theta_1,\dots,\theta_n)$ is finite dimensional over $\mathbb{C}$,
- with $\mathbb{C}\theta_1+\dots+\mathbb{C}\theta_n$ W-stable and carrying the W-representation $V^* \cong \mathbb{C}x_1+\dots+\mathbb{C}x_n$.

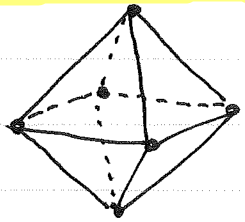
Then $\boxed{\mathrm{Hilb}\left(\left(S/(\underline{\theta})\right)^W, q\right) = \mathrm{Cat}(W,q)}$

EXAMPLE:

W = Weyl group of type B_n or C_n

= $\{n \times n$ signed permutation matrices$\}$

the hyperoctahedral group



acts on

$$\begin{tikzcd} S = \mathbb{C}[x_1,\dots,x_n] & \text{permuting and negating coordinates.} \\ S^W = \mathbb{C}[e_1(x_1^2,\dots,x_n^2), e_2(x_1^2,\dots,x_n^2), \dots, e_n(x_1^2,\dots,x_n^2)] \arrow[u, hook] \end{tikzcd}$$

degrees: $2, 4, \dots, 2n =: h$

Here one can take $\boxed{\underline{\Theta} = (x_1^{2n+1}, \dots, x_n^{2n+1})}$

and $\mathrm{Hilb}\Big(\underset{\substack{\| \\ (\mathbb{C}[x_1,\dots,x_n]/(x_1^{2n+1},\dots,x_n^{2n+1}))^W}}{(S/(\underline{\Theta}))^W}, q\Big) = \dfrac{[2+2n]_q[4+2n]_q \cdots [2n+2n]_q}{[2]_q\,[4]_q \cdots [2n]_q}$

$= \boxed{\begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} = \mathrm{Cat}(B_n, q)}$

EXAMPLE: For $W = \mathfrak{S}_n$ of type A_{n-1} acting irreducibly on $\mathbb{R}^{n-1}$, producing $\underline{\Theta} = (\Theta_1,\dots,\Theta_{n-1})$ is actually subtle! (e.g. Haiman/Kraft 1994 give a clever construction)

But there is an obvious $GL_n(\mathbb{F}_q)$-analogue...

We've seen from Dickson's Theorem that when

$$GL_n(\mathbb{F}_q) \text{ acts on } \underset{\uparrow}{S} = \mathbb{F}_q[x_1, \dots, x_n]$$

one has $S^{GL_n(\mathbb{F}_q)} = \mathbb{F}_q[D_{n,n-1}, \dots, D_{n,1}, D_{n,0}]$

degrees: $q^n - q^{n-1}, \dots, q^n - q^1, q^n - q^0 =: h$

If we define $\underline{\Theta} = (\Theta_1, \dots, \Theta_n)$
$:= (x_1^{q^n}, x_2^{q^n}, \dots, x_n^{q^n})$

then they are

- homogeneous of degree $h+1 \,(= q^n - q^0 + 1)$,
- giving a system of parameters for S,
- with $\mathbb{F}_q \Theta_1 + \mathbb{F}_q \Theta_2 + \dots + \mathbb{F}_q \Theta_n$

$$= \mathbb{F}_q x_1^{q^n} + \mathbb{F}_q x_2^{q^n} + \dots + \mathbb{F}_q x_n^{q^n}$$

$$= \{ (c_1 x_1 + c_2 x_2 + \dots + c_n x_n)^{q^n} : (c_1, \dots, c_n) \in \mathbb{F}_q^n \}$$

a $GL_n(\mathbb{F}_q)$-stable subspace, carrying the representation $V^* = \mathbb{F}_q x_1 + \dots + \mathbb{F}_q x_n$

Actually the same works for any $\underline{\Theta} = (x_1^{q^m}, x_2^{q^m}, \dots, x_n^{q^m})$ with $m \in \{1, 2, \dots\}$

This led us to the following...

CONJECTURE 1 (Lewis-Stanton-R. 2014)

For $G=GL_n(\mathbb{F}_q)$ acting on $S=\mathbb{F}_q[x_1,\dots,x_n]$

and $\underline{\Theta} := (\Theta_1,\dots,\Theta_n) = (x_1^{q^m},\dots,x_n^{q^m})$ with $m\in\{1,2,\dots\}$

$$\mathrm{Hilb}\Big(\underbrace{\big(S/(\underline{\Theta})\big)^{GL_n(\mathbb{F}_q)}}_{\big(\mathbb{F}_q[x_1,\dots,x_n]/(x_1^{q^m},\dots,x_n^{q^m})\big)^{GL_n(\mathbb{F}_q)}}, t\Big) = \sum_{k=0}^{\min(m,n)} t^{(n-k)(q^m-q^k)} \begin{bmatrix} m\\ k\end{bmatrix}_{q,t}$$

the (q,t)-binomials again!

Much evidence exists, but it has only been proven so far when $n\leq 2$ or $m\leq 1$

One piece of evidence is that is consistent with a certain provable CSP triple

$$\Big(\underset{\{P\text{-orbits on } (\mathbb{F}_{q^m})^n\}}{\underset{\|}{X}}\ ,\ X(t)\ ,\ \underset{\mathbb{F}_{q^m}^{\times}}{\underset{\|}{\underset{\langle c\rangle}{\underset{\|}{C}}}} \Big)$$

for $P=\left\{\left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right]\right\} \leq GL_n(\mathbb{F}_q)$, with blocks of sizes $\underbrace{\ }_{k}\ \underbrace{\ }_{n-k}$

simultaneously scaling $(\mathbb{F}_{q^m})^n$

Even more strangely,

using Gorenstein duality in

$$S/(\underline{\Theta}) = \mathbb{F}_q[x_1,\ldots,x_n]/(x_1^{q^m},\ldots,x_n^{q^m})$$

and taking a limit as $m \to \infty$ in

CONJ 1: $\mathrm{Hilb}\left(\left(S/(\underline{\Theta})\right)^{GL_n(\mathbb{F}_q)}, t \right)$

$$= \sum_{k=0}^{\min(m,n)} t^{(n-k)(q^m-q^k)} \begin{bmatrix} m \\ k \end{bmatrix}_{q,t}$$

$$\Big\Downarrow$$ leads to and implies

CONJECTURE 2 (Lewis-Stanton-R. 2014)

The $GL_n(\mathbb{F}_q)$-fixed quotient

$$S_{GL_n(\mathbb{F}_q)} := \mathbb{F}_q[x_1,\ldots,x_n] \Big/ \mathbb{F}_q\text{-span of } \{ f(\underline{x}) - g(f(\underline{x})) \}_{\substack{f \in S \\ g \in GL_n(\mathbb{F}_q)}}$$

has

$$\mathrm{Hilb}(S_{GL_n(\mathbb{F}_q)}, t) = 1 + \frac{t^{n(q-1)}}{1-t^{q-1}} + \frac{t^{n(q^2-1)}}{(1-t^{q^2-1})(1-t^{q^2-q})} + \cdots + \frac{t^{n(q^n-1)}}{(1-t^{q^n-1})\cdots(1-t^{q^n-q^{n-1}})}$$

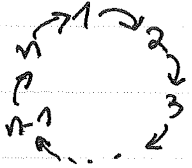
$$= \sum_{k=0}^{n} t^{n(q^k-1)} \Big/ (1-t^{q^k-1})(1-t^{q^k-q})\cdots(1-t^{q^k-q^{k-1}})$$

More analogues – counting chains in NC($\mathfrak{S}_n$)

THEOREM (Hurwitz 1891, Dénes 1959)

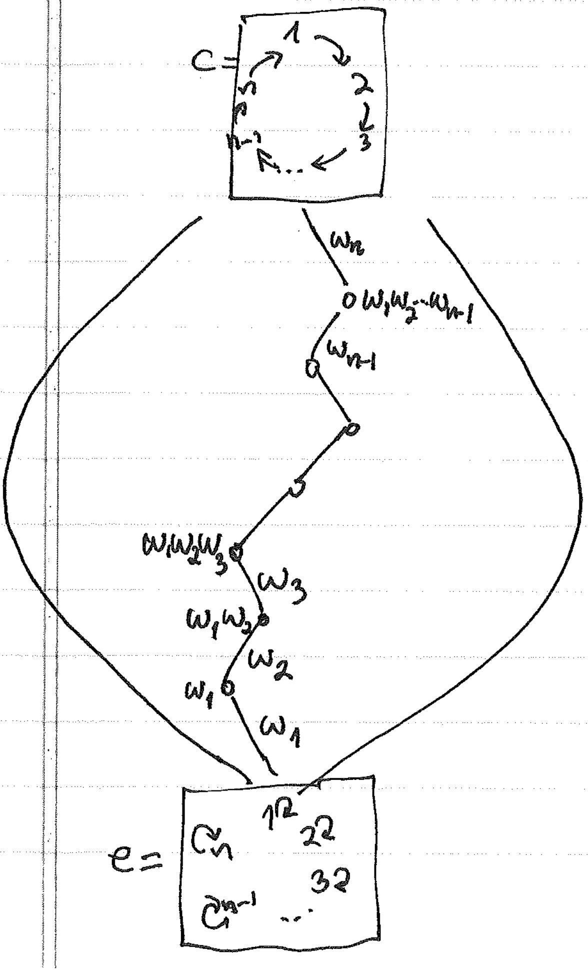
$\#\{$factorizations $c = w_1 w_2 \cdots w_{n-1}$ with w_i = transpositions $(j,k)\}$ $= \boxed{n^{n-2}}$

c // n-cycle

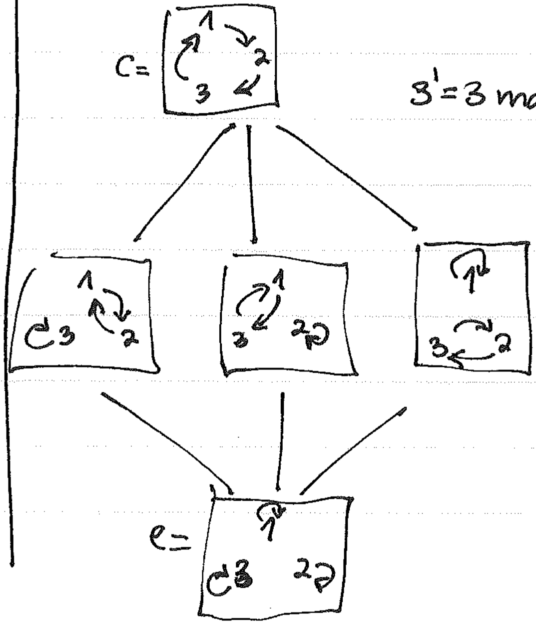


(= # maximal chains in NC($\mathfrak{S}_n$) as a poset)

totally ordered subsets



EXAMPLE: $n=3$

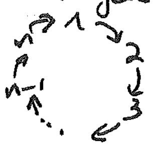


$3^1 = 3$ maximal chains

More generally....

THEOREM (Goulden-Jackson "Cactus formula" 1992)

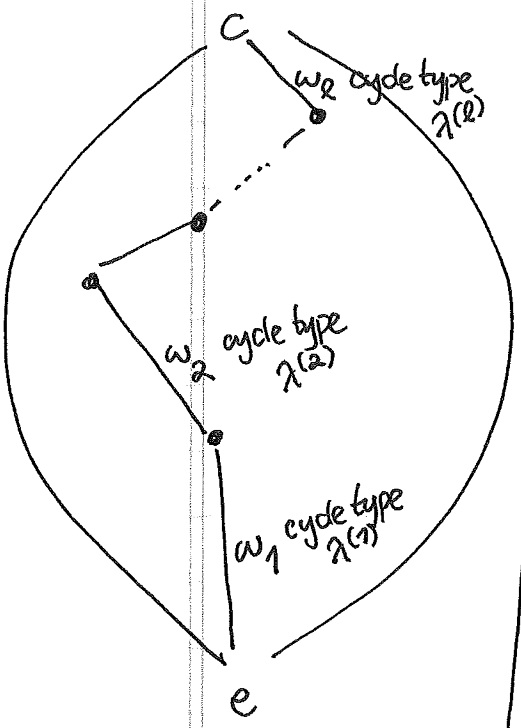
#{factorizations $\underset{\substack{\| \\ n\text{-cycle}}}{c} = w_1 w_2 \cdots w_\ell$ where



- w_i has cycle type partition $\lambda^{(i)}$
- $\sum_i \underbrace{\ell(w_i)}_{\substack{\text{absolute} \\ \text{or reflection} \\ \text{length}}} = n-1$ }

$\ell(w) = n - \#\text{cycles}(w)$
$= \text{codim}(V^w)$

$$= n^{\ell-1} N(\lambda^{(1)}) \cdots N(\lambda^{(\ell)})$$



where $N(\underset{\substack{\| \\ 1^{m_1}2^{m_2}3^{m_2}}}{\lambda}) := \frac{1}{m}\binom{m}{m_1, m_2, m_3 \cdots}$

with $m := m_1 + m_2 + m_3 + \cdots$
$= \#\text{parts of } \lambda$

EXAMPLE: Transpositions (j,k) have cycle type $\underset{\substack{\| \\ \lambda}}{2^1 1^{n-2}}$

so $N(2^1 1^{n-2}) = \frac{1}{n-1}\binom{n-1}{1, n-2} = 1$.

Also they have $\ell((j,k)) = 1$, so $\ell = n-1$.

Hence $n^{\ell-1} N(\lambda^{(1)}) \cdots N(\lambda^{(\ell)}) = n^{(n-1)-1} \cdot 1 \cdot 1 \cdots 1 = n^{n-2}$

EXAMPLE: What if instead of

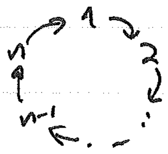
$$\{\text{transpositions}\} = \{\text{cycle type } 2^1 1^{n-2}\}$$

we allow one nontrivial cycle of sizes $\alpha_1, \alpha_2, \ldots$

i.e. cycle types $\lambda^{(1)} = (\alpha_1^1, 1^{n-\alpha_1}),\ \lambda^{(2)} = (\alpha_2^1, 1^{n-\alpha_2}),\ \ldots,\ \lambda^{(\ell)} = (\alpha_\ell^1, 1^{n-\alpha_\ell})$?

The cactus formula still comes out remarkably simple, because $N((\alpha^1, 1^{n-\alpha})) = \frac{1}{n-\alpha+1}\binom{n-\alpha+1}{1,\, n-\alpha} = 1$:

$$\#\Big\{\text{factorizations } c = w_1 w_2 \cdots w_\ell \text{ with } w_i \text{ of cycle type } (\alpha_i^1, 1^{n-\alpha_i}) \text{ and } \sum_{i=1}^{\ell} \underbrace{\ell(w_i)}_{\alpha_i - 1} = n-1\Big\}$$



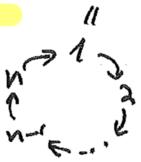
$$= n^{\ell-1}\, \underbrace{N(\alpha_1^1, 1^{n-\alpha_1})}_{1}\, \underbrace{N(\alpha_2^1, 1^{n-\alpha_2})}_{1} \cdots \underbrace{N(\alpha_\ell, 1^{n-\alpha_\ell})}_{1}$$

$$= \boxed{n^{\ell-1}}$$

(independent of the $\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}$!)

By the way, why did Hurwitz care?

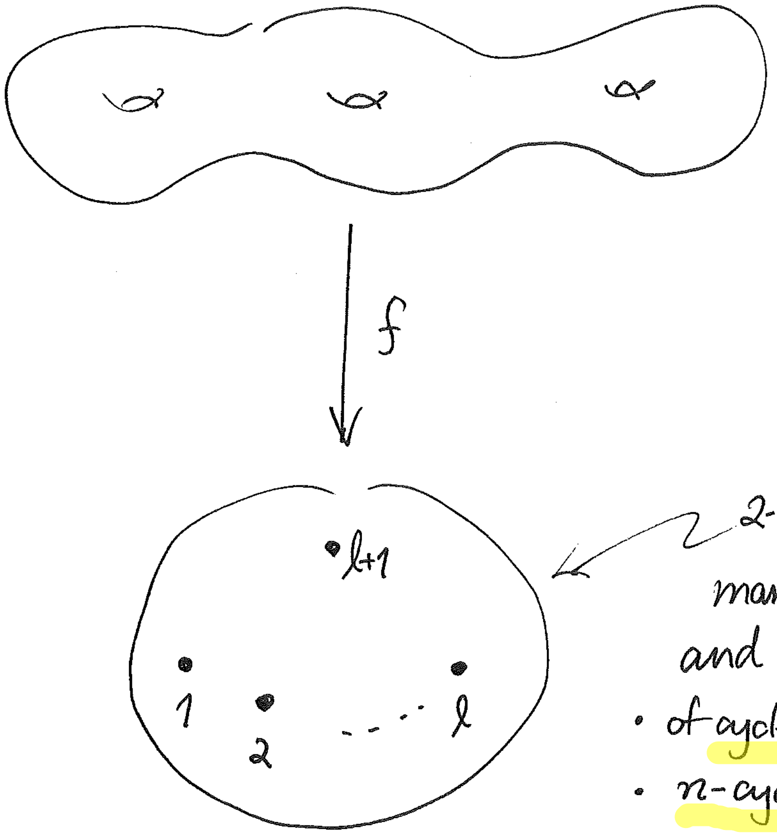
$\#\{$ factorizations $c = \omega_1 \omega_2 \cdots \omega_\ell$ where



- ω_i has cycle type partition $\lambda^{(i)}$
- $\sum_i \ell(\omega_i) = n-1-2g$ $\}$

counts (up to a certain equivalence)

the branched coverings



genus g compact orientable surface

2-sphere with marked branch points and monodromy permutations

- of cycle type $\lambda^{(i)}$ around i
- n-cycle around $\ell+1$

$GL_n(\mathbb{F}_q)$-analogues....

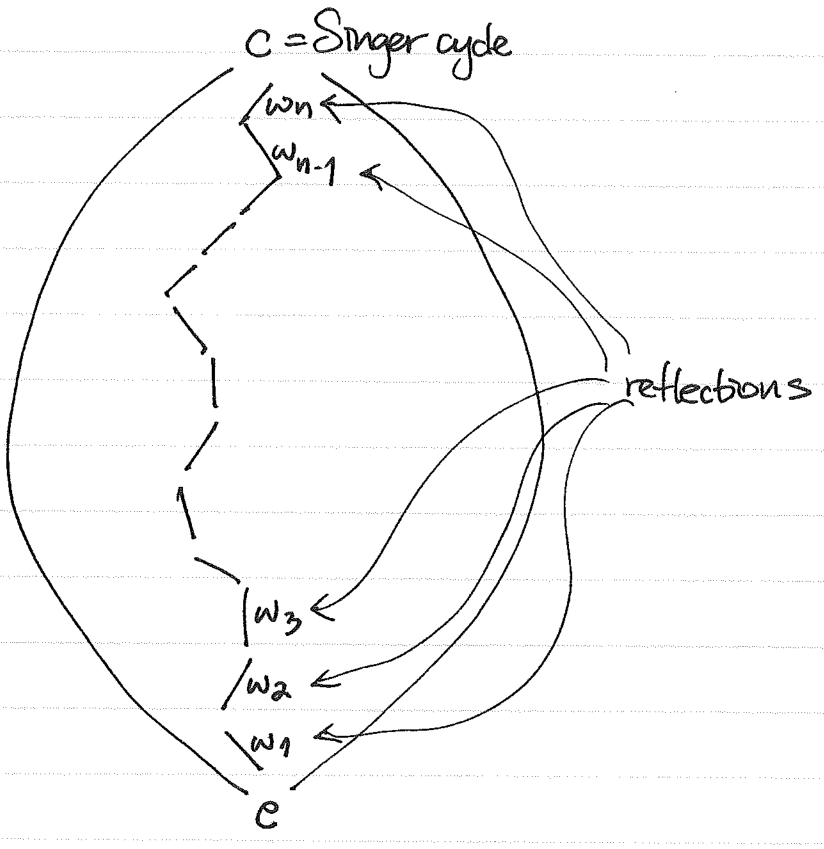
THEOREM (Lewis-Stanton-R. 2014)

$$\#\left\{\text{factorizations } \underset{\substack{\| \\ \text{Singer cycle} \\ \text{in } GL_n(\mathbb{F}_q)}}{c} = w_1 w_2 \cdots w_n \text{ with } w_i \text{ reflections in } GL_n(\mathbb{F}_q)\right\}$$

$$= \boxed{(q^n-1)^{n-1}}.$$

"noncrossing partitions for $GL_n(\mathbb{F}_q)$"?

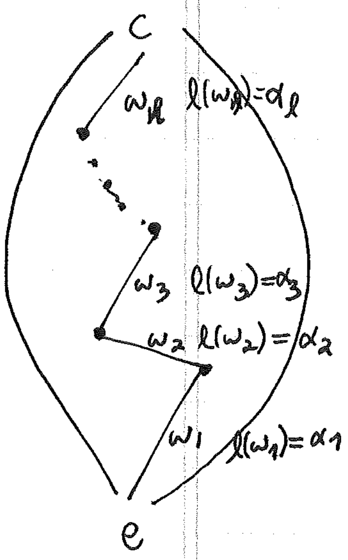
(= # maximal chains in $NC(GL_n(\mathbb{F}_q))$ as a poset)



And more generally ...

THEOREM (Huang-Lewis-R. 2015)

$$\#\left\{\text{factorizations } \underset{\substack{\parallel \\ \text{Singer cycle in } GL_n(\mathbb{F}_q)}}{c} = w_1 w_2 \cdots w_\ell \text{ with } \begin{array}{l} \bullet\ \boxed{\underset{\substack{\parallel \\ \text{absolute or reflection length} \\ \parallel \\ \mathrm{codim}(V^{w_i})}}{\ell(w_i)} = \alpha_i} \\ \bullet\ \sum_{i=1}^{\ell} \underbrace{\ell(w_i)}_{\alpha_i} = n \end{array}\right\}$$



$$= \boxed{(q^n-1)^{\ell-1} \cdot q^{e(\alpha)}}$$

where $e(\alpha_1,\alpha_2,\ldots,\alpha_\ell) := \sum_{i=1}^{\ell} (\alpha_i-1)(n-\alpha_i)$

REMARKS:

- Almost, but not quite, independent of $(\alpha_1,\alpha_2,\ldots,\alpha_\ell)$; still independent of their order
- Very reminiscent of $n^{\ell-1}$?

How to prove these kinds of factorization counts?

The Goulden-Jackson Cactus formula

$$n^{\ell-1} N(\lambda^{(1)}) \cdots N(\lambda^{(\ell)})$$

has (multiple) bijective proofs

PROBLEM: Find a bijective/combinatorial proof for *any* of the $GL_n(\mathbb{F}_q)$-analogues, even the $(q^n-1)^{n-1}$ analogue of n^{n-2}!

We instead relied on a tried-and-true general method:

THEOREM (Frobenius 1896)
In a finite group G,

$$\#\left\{\begin{array}{l}\text{factorizations } c = w_1 w_2 \cdots w_\ell \text{ with} \\ \cdot\ w_i \in K_i \text{ for } i=1,2,\ldots,\ell \text{ for some} \\ \text{fixed conjugacy-closed sets } K_1,\ldots,K_\ell \subseteq G\end{array}\right\}$$

$$= \frac{1}{|G|} \sum_{\substack{\text{irreducible}\\ \text{complex}\\ \text{characters}\\ \chi \text{ of } G}} \deg(\chi) \cdot \chi(c^{-1})\, \tilde{\chi}(K_1) \cdots \tilde{\chi}(K_\ell)$$

where $\tilde{\chi}(K) := \frac{1}{\deg \chi} \sum_{g \in K} \chi(g)$

There are simplifying features in the setting of $\mathfrak{S}_n$ and the cactus formula ....

- When $c = n$-cycle, the irreducible $\mathfrak{S}_n$-character χ^μ indexed by a partition μ of n has

$$\chi^\mu(\underset{n\text{-cycle}}{\underset{\shortparallel}{c}}) = \begin{cases} (-1)^k & \text{if } \mu = \overbrace{\boxed{}}^{n-k}\boxed{\,}\big\}k \\ 0 & \text{otherwise} \end{cases} \qquad k=0,1,\ldots,n-1$$

- $\deg\left(\chi^{\overbrace{\boxed{}}^{n-k}\boxed{\,}\}k}\right) = \binom{n-1}{k}$

Hence Frobenius's THM gives

$$\#\left\{\begin{array}{l}\text{factorizations}\\ c = w_1 w_2 \cdots w_\ell \\ w_i \text{ of cycle type } \lambda^{(i)}\end{array}\right\} = \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\,\tilde{\chi}^{\boxed{}\,k}(\lambda^{(1)})\cdots\tilde{\chi}^{\boxed{}\,k}(\lambda^{(\ell)})$$

- $\tilde{\chi}^{\overbrace{\boxed{}}^{n-k}\boxed{\,}k}(\lambda)$ turns out to agree with a polynomial in k, of degree $n-\#\text{parts}(\lambda)$, with predictable top coefficient.

- Thus $\tilde{\chi}^{\overbrace{\boxed{}}^{n-k}\boxed{\,}k}(\lambda^{(1)})\cdots\tilde{\chi}^{\overbrace{\boxed{}}^{n-k}\boxed{\,}k}(\lambda^{(\ell)}) = f(k)$ for some polynomial $f(k)$ of degree $n-1$ with predictable top coefficient.

~~$\Rightarrow \#\{\text{factorizations}\} = \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k} = \frac{1}{n!}\Delta^{n-1}f(0)$~~

So $f(k)$ is a polynomial of degree $n-1$ in k

and $$\#\{\text{factorizations}\} = \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}f(k)$$

$$= \frac{(-1)^{n-1}}{n!}\underbrace{\Delta^{n-1}f(0)}_{(n-1)^{st}\text{ forward difference of } f \text{ at } x=0}$$

$$\Delta^0 f(0) = f(0)$$

$$\Delta^1 f(0) = f(1) - f(0)$$

$$\Delta^1(\Delta^1 f(0)) = \Delta^2 f(0) = (f(2)-f(1)) - (f(1)-f(0))$$

$$= f(2) - 2f(1) + f(0)$$

$$\Delta^3 f(0) = \binom{3}{3}f(3) - \binom{3}{2}f(2) + \binom{3}{1}f(1) - \binom{3}{0}f(0)$$

$$\vdots$$

EASY FACT: (EXERCISE!)

$$\Delta^N f(0) = \begin{cases} 0 & \text{if } \deg(f) < N \\ N!\left(\begin{matrix}\text{leading}\\ \text{coefficient}\\ \text{of } f\end{matrix}\right) & \text{if } \deg(f) = N \end{cases}$$

Thus $\#\{\text{factorizations}\} = \frac{(-1)^{n-1}}{n!}\Delta^{n-1}f(0)$

is easily computed from the known leading coefficient of f in terms of $\lambda^{(1)}, \ldots, \lambda^{(l)}$. $\Rightarrow$ Cactus formula ∎

The $GL_n(\mathbb{F}_q)$-analogue uses $GL_n(\mathbb{F}_q)$-characters, $\begin{bmatrix}n-1\\k\end{bmatrix}_q$, q-differences, ...

59

THANKS for

listening !

And THANK YOU

INdAM

&

CRM !