BOUNDARY-CONFORMING DISCONTINUOUS GALERKIN METHODS VIA EXTENSIONS FROM SUBDOMAINS

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Abstract. A new way of devising numerical methods is introduced whose distinctive feature is the computation of a finite element approximation only in a polyhedral subdomain D of the original, possibly curved-boundary domain. The technique is applied to a discontinuous Galerkin method for the one-dimensional diffusion-reaction problem. Sharp a priori error estimates are obtained which identify conditions, on the subdomain D and the discretization parameters of the discontinuous Galerkin method, under which the method maintains its original optimal convergence properties. The error analysis is new even in the case in which $\mathsf{D} = \Omega$. It allows to see that the uniform error at any given interval is bounded by an interpolation error associated to the interval plus a significantly smaller error of a global nature. Numerical results confirming the sharpness of the theoretical results are displayed. Also, preliminary numerical results illustrating the application of the method to two-dimensional second-order elliptic problems are shown.

1. Introduction. This is the first of a series of papers in which we introduce and study a new way of defining approximations to solutions of various partial differential equations. Its distinctive feature is the use of finite element approximations defined on elements with *flat* boundaries even when the original domain Ω has *non piecewise-flat* boundaries. Thus only a *polyhedral subdomain* D of the original domain Ω is actually triangulated. This capability is very convenient from the computational point of view since, for example, we can restrict ourselves to taking D as being amenable to uniform triangulation.

Note that to be able to define the finite element approximation on the subdomain D, we must be able to *transfer* the data on the boundary of Ω into that of D. This is done by the introduction of a suitable *extension* of the finite approximation to be computed into the set $\Omega \setminus \mathsf{D}$. The definition of the extension and the way of using it to set the boundary conditions on D are the crucial and novel features of this new technique. If this is done in such a way that the approximate solution converges to the exact one in the *whole* original domain Ω, we say that the resulting method is *boundary-conforming*.

In this paper, we carry out what could be considered to be the necessary first step in the development of boundary-conforming finite element methods, and consider the application of this approach to hybridizable discontinuous Galerkin (HDG) methods, see [4], for the one-dimensional model problem

$$c\,q + u' = 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$q' + d\,u = f \quad \text{in } \Omega, \tag{1.1b}$$

$$u = u_D \quad \text{on } \partial\Omega, \tag{1.1c}$$

where $\Omega = (0, 1)$. We provide a detailed and thorough theoretical analysis which puts in firm mathematical ground this new approach. Indeed, by using a new technique for the a priori analysis of finite element methods, we find conditions, on the distance

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between D and Ω and on some parameters of the HDG method, under which the orders of convergence of the approximation, in various norms, remain the same as those for the case $\mathsf{D} = \Omega$.

Roughly speaking, we show that this is the case, for the pointwise error in both variables, if the distance between D and Ω is of order h. In particular, if we use uniform meshes of step size h and take polynomials of degree k for the approximations for both unknowns, we obtain that the rate of convergence of the pointwise error in both variables is of order

$$\left(\frac{\mathsf{c}\,h}{k+1}\right)^{k+1}.$$

This result is consistent with the L^2-estimates obtained for HDG methods in [3], for a special HDG method, and in [5], for a general class of DG methods including HDG methods.

For the error in negative-order norms (as well as for the error in the so-called numerical traces of the DG method), we find that there is a loss of accuracy which, however, can be recovered by simply increasing the polynomial degree of the approximation at only two intervals, namely, those touching the boundary of D. In particular, if we use uniform meshes of step size h and take polynomials of degree k for the approximations for both unknowns, we obtain that rate of convergence of the error in the $-k$-negative-order norm and in the error of the numerical traces is of order

$$h\,\left(\frac{\mathsf{c}\,h}{k+1}\right)^{k+1}.$$

However, we show that it is enough to rise the polynomial approximation of the two intervals touching the border of D from k to $2\,k-1$ to force the rate of convergence to be of order

$$\left(\frac{\mathsf{c}\,h}{k}\right)^{k}\left(\frac{\mathsf{c}\,h}{k+1}\right)^{k+1} + h\,\left(\frac{\mathsf{c}\,h}{2\,k}\right)^{2\,k}.$$

This is consistent with the results in [1], where the rate of convergence of the numerical fluxes of a wide class of DG methods for one-dimensional convection-diffusion problems was shown to be of order $h^{2\,k+1}$.

Let us also point out that our analysis allows us to determine how the quality of the approximation in any given element is affected by that of the other elements. Roughly speaking, if we take for simplicity $\mathsf{D} = \Omega$, we show that if k_ℓ is the polynomial degree of the approximation in the ℓ-th element, then the pointwise error of both variables in the j-th element are of order

$$\left(\frac{\mathsf{c}\,h}{\min\{k_j, 2\,k\}+1}\right)^{\min\{k_j,2\,k\}+1},$$

where h is the maximum size of the elements and k is the smallest of the polynomial degrees k_ℓ. This means that we can increase the quality of the approximation in the j-th element by simply raising k_j up to $2\,k$.

The organization of the the paper is as follows. In Section 2, we introduce the method and state and briefly discuss our theoretical results. Their proofs are given in Section 3; they use key auxiliary results proved in Section 4 and in the Appendix. In

Section 5, we present numerical results confirming our theoretical predictions. We end in Section 6 by commenting on extensions and by displaying a preliminary numerical experiment showing that the extension of the method to the two-dimensional case works well.

2. The main result.

In this section, we describe the new numerical method and then state and briefly discuss our main results.

2.1. The main idea.

Our approach is based on a simple rewriting of the original boundary-value problem (1.1), namely,

$$c\,q + u' = 0 \qquad \text{in } \mathsf{D} \tag{2.1a}$$

$$q' + d\,u = f \qquad \text{in } \mathsf{D}, \tag{2.1b}$$

$$u(a) = u_D(0) - \int_0^a c(s)\,q(s)\,ds, \tag{2.1c}$$

$$u(b) = u_D(1) + \int_b^1 c(s)\,q(s)\,ds. \tag{2.1d}$$

Here, the function q on $(0,a)\cup(b,1)$ is given by

$$q(x) := \begin{cases} q(a^-) - \int_x^a (f(s) - d(s)\,u(s))\,ds & \text{for } x \in (0,a), \\ q(b^+) + \int_b^x (f(s) - d(s)\,u(s))\,ds & \text{for } x \in (b,1), \end{cases} \tag{2.2a}$$

where

$$u(x) := \begin{cases} u_D(0) - \int_0^x c(s)\,q(s)\,ds & \text{for } x \in (0,a), \\ u_D(1) + \int_x^1 c(s)\,q(s)\,ds & \text{for } x \in (b,1). \end{cases} \tag{2.2b}$$

The method is now defined in two steps. In the first, we approximate (q,u) on the set $(0,a)\cup(b,1)$ by a discrete version of equations (2.2) given in terms of the approximate solution defined *on the domain* D; this is the *extension* of the approximate solution from the subdomain D into the domain Ω we talked about in the Introduction. In a second step, we discretize the boundary-value problem on D given by the equations (2.1).

2.2. The numerical method.

To define our method, we need to introduce some notation. We denote by $\mathcal{T}_h$ a triangulation of D, that is, $\mathcal{T}_h$ is the set of intervals $\{I_j := (x_{j-1}, x_j), j = 1,\ldots,N\}$, where $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$. For each interval $I_j \in \mathcal{T}_h$, we denote its length by h_j and define its outward unit normal $n_{I_j}(x_j) = 1$ and $n_{I_j}(x_{j-1}) = -1$; if there is no confusion, instead of n_{I_j} we simply write n.

We are now ready to define the numerical method. As usual, we take the restriction of the approximate solution (q_h, u_h) to the subdomain D in the space $V_h \times W_h$ where

$$V_h = W_h = \{w \in L^2(\mathcal{T}_h) : \; w|_{I_j} \in \mathcal{P}^{k_j}(I_j)\, j = 1,\ldots,N\},$$

where $\mathcal{P}^{k_j}(I_j)$ denotes the space of polynomials of degree at most $k_j \geq 0$ defined on the set I_j.

In $(0,a)\cup(b,1)$, the approximate solution (q_h, u_h) is then taken to be of the form

$$q_h(x) := \begin{cases} q_h|_{I_1}(x) & \text{for } x\in(0,a),\\ q_h|_{I_N}(x) & \text{for } x\in(b,1),\end{cases} \tag{2.3a}$$

where

$$u_h(x) := \begin{cases} u(0) - \int_0^x c(y)\, q_h(y)\, dy & \text{for } x\in(0,a),\\ u(1) + \int_x^1 c(y)\, q_h(y)\, dy & \text{for } x\in(b,1).\end{cases} \tag{2.3b}$$

Note that these equations are approximations of the corresponding equations (2.2). Note also that q_h on (0,a) is a polynomial of degree k_1 and that q_h on (b,1) is a polynomial of degree k_N. In contrast, u_h is not necessarily a polynomial on $(0,a)$ or $(b,1)$.

The approximate solution on D is determined by requiring that is satisfies a weak version of equations (2.1), namely,

$$(c\, q_h, v)_{\mathcal{T}_h} = (u_h, v')_{\mathcal{T}_h} - \langle \widehat{u}_h\,,\, vn\rangle_{\partial\mathcal{T}_h}, \tag{2.4a}$$

$$-(q_h, w')_{\mathcal{T}_h} + \langle \widehat{q}_h\,,\, wn\rangle_{\partial\mathcal{T}_h} + (d\, u_h, w)_{\mathcal{T}_h} = (f, w)_{\mathcal{T}_h}, \tag{2.4b}$$

$$\widehat{u}_h(a) = u_D(0) - \int_0^a c(x)\, q_h(x)\, dx, \tag{2.4c}$$

$$\widehat{u}_h(b) = u_D(1) + \int_b^1 c(x)\, q_h(x)\, dx, \tag{2.4d}$$

for all $(v,w)\in V_h\times W_h$. Here, we are using the notation

$$(\varphi,\psi)_{\mathcal{T}_h} := \sum_{I_j\in\mathcal{T}_h} (\varphi,\psi)_{I_j} \quad \text{and} \quad \langle \varphi\,,\,\psi\, n\rangle_{\partial\mathcal{T}_h} := \sum_{I_j\in\mathcal{T}_h} \langle \varphi\,,\,\psi\, n\rangle_{\partial I_j}.$$

Note that the Dirichlet boundary conditions are *transferred* into the border of the subdomain D by the extension of (q_h, u_h) defined by (2.3).

To complete the definition of the method, it remains to define the numerical trace for the potential, $\widehat{u}_h$, on the interior nodes and the numerical trace for the flux, $\widehat{q}_h$, on all the nodes. On the interior nodes, we take

$$\widehat{u}_h = \frac{\tau^+\, u_h^+ + \tau^-\, u_h^-}{\tau^+ + \tau^-} + \frac{1}{\tau^+ + \tau^-}(q_h^- - q_h^+), \tag{2.5a}$$

$$\widehat{q}_h = \frac{\tau^-\, q_h^+ + \tau^+\, q_h^-}{\tau^+ + \tau^-} + \frac{\tau^+\,\tau^-}{\tau^+ + \tau^-}(u_h^- - u_h^+), \tag{2.5b}$$

and,

$$\widehat{q}_h(a) = q_h(a^+) + \tau(a^+)\,(u_h(a^+) - \widehat{u}_h(a)), \tag{2.5c}$$

$$\widehat{q}_h(b) = q_h(b^-) - \tau(b^-)\,(u_h(b^-) - \widehat{u}_h(b)). \tag{2.5d}$$

Here, the function τ is any positive function defined on the set $\{\partial I_j\}_{j=1,\ldots,N}$. Note that τ is double-valued on the internal nodes.

Using the fact that τ is positive, a simple algebraic manipulation shows that we have,

$$\widehat{q}_h(x_j^+) = q_h(x_j^+) + \tau(x_j^+)\,(u_h(x_j^+) - \widehat{u}_h(x_j)), \tag{2.6a}$$

$$\widehat{q}_h(x_{j-1}^-) = q_h(x_{j-1}^-) - \tau(x_{j-1}^-)\,(u_h(x_{j-1}^-) - \widehat{u}_h(x_{j-1})). \tag{2.6b}$$

for $j = 1, \ldots, N$. Note that when $a = 0$ and $b = 1$, the methods just described reduce to the HDG methods introduced in [4]. They are known to be well defined provided τ is a positive function; see [4].

2.3. The main assumptions. To state our estimates for the error

$$(e_u, e_q) := (u - u_h, q - q_h),$$

we need to specify the main assumptions under which we prove the estimates.

To state them, we need to introduce some notation. We set

$$I_1^{ext} := (0, a) \quad \text{and} \quad I_N^{ext} := (b, 1), \tag{2.7a}$$

and denote their length by h_1^{ext} and h_N^{ext}, respectively. We also set,

$$\overset{\star}{I}_1 := (0, x_1) \quad \text{and} \quad \overset{\star}{I}_N := (x_{N-1}, 1), \tag{2.7b}$$

and define $\overset{\star}{\mathcal{T}}_h := \cup_{\nu=1,N} \overset{\star}{I}_\nu$. We are now ready to introduce our main assumptions.

First assumption: On the regularity of c and d. We assume that

$$\text{the functions } c \text{ and } d \text{ are analytic on } \Omega. \tag{2.8}$$

This assumption guarantees that the exact solution and the Green's functions associated to our elliptic model problem are as smooth desired. In this way, the already complex error analysis we present here is not hindered by approximability issues. Of course, our error analysis technnique can still be used for less regular functions c and d.

Second assumption: On the distance between D and Ω. We assume that there is a positive real number r such that

$$r_\nu := \frac{h_\nu^{ext}}{h_\nu} \leq \mathsf{r} \qquad \text{for } \nu = 1, N. \tag{2.9}$$

Recall that h_ν^{ext} is the length of the interval I_ν^{ext} and that h_ν is the length of the interval I_ν for $\nu = 1, N$. In other words, we assume that $a \leq \mathsf{r}\, h_1$ and that $(1 - b) \leq \mathsf{r}\, h_N$. In particular, if all the elements of the computational subdomain $\mathcal{T}_h$ have size h and r is an integer, we are assuming the distance between the border of the computational subdomain D and that of the exact domain Ω is at most the size of r elements.

Third assumption: On bounds on the size of τ. We assume that

$$\max_{1 \leq j \leq N} \theta_j \leq \theta, \tag{2.10a}$$

$$\max_{1 \leq j \leq N} \eta_j \leq \eta, \tag{2.10b}$$

where

$$\theta_j := 1/(\tau(x_{j-1}^+) + \tau(x_j^-)), \tag{2.10c}$$

$$\eta_j := \tau(x_{j-1}^+)\,\tau(x_j^-)/(\tau(x_{j-1}^+) + \tau(x_j^-)), \tag{2.10d}$$

for some positive real numbers θ and η bigger or equal to one. As was recently proven in [5], see also [3], this choice of stabilization parameters ensures optimal convergence orders in L^2 of the approximation to both the potential u and its flux q.

Fourth assumption: On the size of h_j and k_j. The fourth and last assumption we take is related with the size of elements of the computational subdomain $\mathcal{T}_h$, h_ℓ, and the polynomials degrees k_ℓ, for $\ell = 1, \ldots, N$. It is written in terms of quantities we define next.

We begin by introducing the following auxiliary function:

$$\mathsf{K}(h; p, s) := \begin{cases} \left(\frac{\pi h}{4}\right)^{\min\{s,p\}} \ \frac{(p-\min\{s,p\})!}{p!} \sqrt{p} & \text{if } p > 0, \\ 1 & \text{if } p = 0. \end{cases} \tag{2.11}$$

It appears in most of our approximation results; see Proposition 4.1. Next, we set, for $j = 1, \ldots, N$,

$$\underline{\kappa}_j := \mathsf{K}(h_j; k_j + 1, 1), \tag{2.12a}$$

$$\kappa_j := \mathsf{K}(h_j; k_j + 1, k_j + 1), \tag{2.12b}$$

and, for $\nu = 1$ and $\nu = N$,

$$\kappa_\nu^{ext} := \mathsf{K}((1 + r_\nu)\, h_\nu; k_\nu + 1, k_\nu + 1), \tag{2.12c}$$

where r_ν is given by (2.9). Finally, we set

$$\kappa_{\rho,j} := \rho_j^{3\,k_j}\ \kappa_j, \quad \kappa_{\rho,\nu}^{ext} := \ \rho_\nu^{3\,k_\nu}\ \kappa_\nu^{ext}, \tag{2.12d}$$

where

$$\rho_j := \begin{cases} 1 + 2\, r_\nu + 2\,\sqrt{r_\nu(1 + r_\nu)} & \text{for } j = 1 \text{ or } j = N, \\ 1 & \text{otherwise.} \end{cases} \tag{2.12e}$$

We are now ready to state our last assumption. We assume that

$$\kappa := \max_{1 \le j \le N} \{\underline{\kappa}_j, \kappa_j\} + \max_{\nu=1,N} \kappa_{\rho,\nu}^{ext} \quad \text{is small enough.} \tag{2.13}$$

Note that, since

$$\underline{\kappa}_j = \frac{\pi\, h_j}{4\,\sqrt{k_j + 1}} \quad \text{and} \quad \kappa_j \le \left(\frac{\pi\, e\, h_j}{4\,(k_j + 1)}\right)^{k_j+1},$$

after an application of Stirling's formula, we see that the above smallness condition can be always satisfied. In particular, in the so-called h-version of the method, is is enough to render the number $\max_{1 \le j \le N} h_j$ small enough. In the so-called p-version of the method, it is enough to render the number $\min_{1\le j\le N} k_j$ big enough. Note also that since

$$\kappa^{ext}_{\rho,\nu} \le \rho_\nu^{3\,k_\nu} \; \left(\frac{\pi\, e\,(1+r_\nu)\, h_\nu}{4\,(k_\nu+1)}\right)^{k_\nu+1},$$

the effect of taking a *bigger* h^{ext}_ν, which results in bigger parameters ρ_ν and r_ν, can be easily counterbalanced by raising the polynomial degree k_ν.

Note also that the above assumption *implies* the three *smallness* conditions that must be satisfied in order for our error analysis to hold; see the inequalities (4.3), (4.4) and (4.9).

Next, we state and briefly discuss our main results.

2.4. The a priori error estimates. To state our results, we need to introduce a couple of seminorms. We set $|\,\phi\,|_{W^{k,\infty}(I)} := \;|\,\phi^{(k)}\,|_{L^\infty(I)}$ and

$$\begin{aligned}
|(u,q)|_j &:= \max\{|\,u\,|_{W^{k_j+1,\infty}(I_j)}, |\,q\,|_{W^{k_j+1,\infty}(I_j)}\},\\
\|\!|\,(u,q)\,\|\!| &:= \max\{\max_{1\le j\le N} |\,u\,|_{W^{k_j+1,\infty}(I_j)},\\
&\qquad \max_{2\le j\le N-1} |\,q\,|_{W^{k_j+1,\infty}(I_j)}, \max_{\nu=1,N} |\,q\,|_{W^{k_\nu+1,\infty}(\overset{\star}{I}_\nu)}\}.
\end{aligned}$$

We are also going to use the expression

$$\boldsymbol{\kappa} := \max_{1\le j\le N}\{\mathsf{K}(h_j;k_j,k_j)\,\kappa_j\} + \max_{\nu=1,N}\{\kappa^{ext}_{\rho,\nu}\, h^{ext}_\nu\}.$$

We are now ready to obtain bounds for the error $(e_u,e_q) = (u-u_h, q-q_h)$.

Theorem 2.1 (Local uniform error estimates). *Assume that the four assumptions in §2.3 hold. Then, for $j = 1,\dots,N$,*

$$\begin{aligned}
\|\,e_u\,\|_{L^\infty(I_j)} &\le C\;\kappa_j\;\Lambda_{u,j} + C\;\boldsymbol{\kappa}\;\|\!|\,(u,q)\,\|\!|,\\
\|\,e_q\,\|_{L^\infty(I_j)} &\le C\;\kappa_j\;\Lambda_{q,j} + C\;\boldsymbol{\kappa}\;\|\!|\,(u,q)\,\|\!|,
\end{aligned}$$

and for $\nu = 1, N$,

$$\begin{aligned}
\|\,e_u\,\|_{L^\infty(I^{ext}_\nu)} &\le C\;\kappa\,\kappa^{ext}_{\rho,\nu} + C\;\boldsymbol{\kappa}\;\|\!|\,(u,q)\,\|\!|,\\
\|\,e_q\,\|_{L^\infty(I^{ext}_\nu)} &\le C\,\Lambda_{q,\nu}\;\kappa^{ext}_{\rho,\nu} + C\;\boldsymbol{\kappa}\;\|\!|\,(u,q)\,\|\!|,
\end{aligned}$$

for some constant C. Here

$$\begin{aligned}
\Lambda_{u,j} &:= |\,u\,|_{W^{k_j+1,\infty}(I_j)} + \max\{\underline{\kappa}_j, \kappa_j\}\;\Big(|\,u\,|_{W^{k_j+1,\infty}(I_j)} + |\,q\,|_{W^{k_j+1,\infty}(I_j)}\Big),\\
\Lambda_{q,j} &:= |\,q\,|_{W^{k_j+1,\infty}(I_j)} + \max\{\underline{\kappa}_j, \kappa_j\}\;\Big(|\,u\,|_{W^{k_j+1,\infty}(I_j)} + |\,q\,|_{W^{k_j+1,\infty}(I_j)}\Big),
\end{aligned}$$

and

$$\Lambda_{q,\nu} := |\,q\,|_{W^{k_\nu+1,\infty}(\overset{\star}{I}_\nu)} + \underline{\kappa}_\nu\;\big(|\,u\,|_{W^{k_\nu+1,\infty}(I_\nu)} + |\,q\,|_{W^{k_\nu+1,\infty}(I_\nu)}\big).$$

Let us briefly discuss some consequences of this result:

• First of all, let us point out that the above theorem gives a condition for the solution of the numerical method (2.4) to exist and be unique. Indeed, since the equations given by (2.4) generate a square system, the existence and uniqueness of the approximation solution is equivalent to proving that the only solution of the equations (2.4) with zero data is the trivial one. Since the exact solution in such a case is the zero solution, the above error estimates guarantee that the approximate solution is also the trivial one.

• In the case in which $h_j = h$ and $k_j = k$ for $j = 1, \ldots, N$, the above theorem states that $\| e_u \|_{L^\infty(0,1)}$ and $\| e_u \|_{L^\infty(0,1)}$ are of order

$$\left(\frac{\mathsf{c}\, h}{k+1}\right)^{k+1},$$

for small enough $h/\sqrt{k+1}$. Note that this holds true regardless of the values of the parameters r_ν for $\nu = 1$ and $\nu = N$; of course, c depends on those parameters.

• Note how the bound of each of the errors e_u and e_q in a given interval I_j, for $j = 1, \cdots, N$, is equal to an interpolation error in that interval plus a global error which is, typically, significantly smaller. For example, in the previous case, the interpolation error is of order

$$\left(\frac{\mathsf{c}\, h}{k+1}\right)^{k+1},$$

whereas the global error is of order

$$\left(\left(\frac{\mathsf{c}\, h}{k}\right)^{k} + \max_{\nu=1,N} r_\nu\, h\right)\left(\frac{\mathsf{c}\, h}{k+1}\right)^{k+1}.$$

We see that the global error becomes remarkably smaller than the interpolation error as h goes to zero, since r_ν is bounded for $\nu = 1$ and $\nu = N$ by our second assumption (2.9).

• Although something similar happens to the errors e_u and e_q in the intervals I_ν^{ext} for $\nu = 1$ and $\nu = N$, the estimates for the errors e_u and e_q in those intervals are different because of the way the approximate solution was extended to the set $\Omega \setminus \mathsf{D}$. In particular, since e_u is an integral of e_q therein, see the equations (2.2b) and(2.3b), it is reasonable to expect the error e_u to converge *faster* than the error e_q. This is exactly what happens since, for the example we have been working with, the error e_u is of order

$$\left(\frac{h}{\sqrt{k+1}}\right)\left(\frac{\mathsf{c}\, h}{k+1}\right)^{k+1}.$$

• Note also that this theorem tells us under what conditions we can decrease the errors e_u and e_q in any given interval I_j by simply increasing the polynomial degree k_j therein. Let us illustrate this important point. If we modify the example we have been considering by simply taking k_j to be possibly bigger than k, and by setting $a = 0$ and $b = 1$, we see that the above theorem states that the errors e_u and e_q in the interval I_j are of order

$$\left(\frac{\mathsf{c}\, h}{\min\{k_j, 2\,k\} + 1}\right)^{\min\{k_j, 2\,k\}+1}.$$

Thus, increasing the polynomial degree k_j does increase rate of convergence when k_j goes from k to $2\,k$. Beyond that value, we do not expect to see any improvement in the approximation since then the global error begins to dominate the local error.

• When $d = 0$, c is a constant, $k_j \geq 1$ for $j = 1, \ldots, N$ *and* $a = 0, b = 1$, the product $C\,\boldsymbol{\kappa}$ can be replaced by zero. The error of the method is thus bounded by the local approximation error only. This is a well known result for the continuous Galerkin method which reflects the fact that in this case the errors at the nodes are identically equal to zero. In turn, this is a consequence of having the corresponding Green's function in the finite element space *and* not having any errors in the Dirichlet boundary conditions. A similar argument can be made in our case. Let us note, however, that if $a > 0$ or $b < 1$, errors in the Dirichlet boundary conditions are introduced and we canno longer replace $C\,\boldsymbol{\kappa}$ by zero; thus, the global component of the error reappears.

Next, we give a result on the convergence of the numerical traces.

THEOREM 2.2 (Superconvergence of numerical traces). *Assume that the four assumptions in §2.3 hold. Then*

$$\| \widehat{e}_u \|_{L^\infty(\{x_j\}_{j=0}^N)} \leq C\,\boldsymbol{\kappa} \,|\!|\!|\, (u,q) \,|\!|\!|,$$
$$\| \widehat{e}_q \|_{L^\infty(\{x_j\}_{j=0}^N)} \leq C\,\boldsymbol{\kappa} \,|\!|\!|\, (u,q) \,|\!|\!|,$$

for some constant C.

Note that, in the case in which $h_j = h$ and $k_j = k$ for $j = 1, \ldots, N$, the order of convergence of $\| \widehat{e}_u \|_{L^\infty(\{x_j\}_{j=0}^N)}$ and $\| \widehat{e}_q \|_{L^\infty(\{x_j\}_{j=0}^N)}$ is

$$\left(\left(\frac{\mathsf{c}\,h}{k} \right)^k + \max_{\nu=1,N} r_\nu\, h \right) \left(\frac{\mathsf{c}\,h}{k+1} \right)^{k+1}.$$

When $a = 0$ and $b = 1$, this gives the order of convergence of $h^{2\,k+1}$ obtained in [1] within the framework of a convection-diffusion model problem. However, in the general case $a > 0$ and $b < 1$, if we take $k_1 = k_N = k^{ext}$ since then $\boldsymbol{\kappa}$ is of the order of

$$\left(\frac{\mathsf{c}\,h}{k} \right)^k \left(\frac{\mathsf{c}\,h}{k+1} \right)^{k+1} + \max_{\nu=1,N} r_\nu\, h \left(\frac{\mathsf{c}\,h}{k^{ext}-1} \right)^{k^{ext}+1},$$

which means that if we have to take $k^{ext} = 2\,k - 1$ to recover the rate of convergence of the case $a = 0$ and $b = 1$.

When $d = 0$, c is a constant, $k_j \geq 1$ for $j = 1, \ldots, N$ *and* $a = 0, b = 1$, the product $C\,\boldsymbol{\kappa}$ can be replaced by zero in the above result. Hence, the errors in the numerical fluxes are identically equal to zero; see [1]. Moreover, if $a > 0$ or $b < 1$, errors in the Dirichlet boundary conditions are introduced and $C\,\boldsymbol{\kappa}$ *cannot* be replaced by zero anymore.

Finally, we give bounds on the errors in negative-order norms. The result will be stated in terms of the quantity

$$\boldsymbol{\kappa}(s) := \max_{1 \leq j \leq N} \{ \mathsf{K}(h_j; k_j, s)\, \kappa_j \} + \max_{\nu=1,N} \{ \kappa_{\rho,\nu}^{ext}\, h_\nu^{ext} \}.$$

THEOREM 2.3 (Negative-order norm error estimates). *Assume that the four assumptions in §2.3 hold. Then*

$$\| e_u \|_{H^{-s}(0,1)} \leq C\,\boldsymbol{\kappa}(s) \,|\!|\!|\, (u,q) \,|\!|\!|,$$
$$\| e_q \|_{H^{-s}(0,1)} \leq C\,\boldsymbol{\kappa}(s) \,|\!|\!|\, (u,q) \,|\!|\!|,$$

for any integer $s > 0$, for some constant C.

Note that the norms of the errors $\| e_u \|_{H^{-k}(0,1)}$ and $\| e_q \|_{H^{-k}(0,1)}$ converge with the same rate than the numerical traces. Thus, to obtain their optimal rate of convergence in the example under consideration, we have to take $k^{ext} = 2\,k - 1$.

3. Proofs of the main theorems. In this section, we provide detailed proofs of our main results, Theorems 2.1, 2.3, and 2.2. Roughly speaking, the idea of the proof is to express the error $(e_u, e_q) = (u - u_h, q - q_h)$ as the sum of a *local* interpolation error and some *compact* operators acting of both the interpolation error and the actual error (e_u, e_q). This error representation will allow us to prove all our results.

As usual, we base our analysis on the error equations, namely,

$$(c\,e_q, v)_{\Omega_h} - (e_u, v')_{\Omega_h} + < \widehat{e}_u, v \cdot n >_{\partial\,\Omega_h} = 0, \tag{3.1a}$$

$$-(e_q, w')_{\Omega_h} + < \widehat{e}_q \cdot n, w >_{\partial\,\Omega_h} + (d\,e_u, w)_{\Omega_h} = 0, \tag{3.1b}$$

$$\widehat{e}_u(a) = -\int_0^a c(x) e_q(x)\,dx \tag{3.1c}$$

$$\widehat{e}_u(b) = \int_b^1 c(x) e_q(x)\,dx, \tag{3.1d}$$

for all $(v, w) \in V_h \times W_h$.

3.1. The error representation. To obtain our error representation formula, we proceed in two steps. In the first, we use the error equations to identify an auxiliary function $(\widehat{\varepsilon}_u, \widehat{\varepsilon}_q)$ whose residual is $(-c\,S_q, -d\,S_u)$, where (S_u, S_q) is nothing but $(e_u, e_q) - (\widehat{\varepsilon}_u, \widehat{\varepsilon}_q)$. In the second step, we show that $(\widehat{\varepsilon}_u, \widehat{\varepsilon}_q)$ can be expressed as a *compact* operator applied to the function (S_u, S_q). In this way, the error is expressed solely in terms of the function (S_u, S_q) which, thanks to the structure of the numerical scheme, will turn out to be small.

Step 1: The function $(\widehat{\varepsilon}_u, \widehat{\varepsilon}_q)$ and its residual $(-c\,S_q, -d\,S_u)$. We begin with the following result.

LEMMA 3.1. *Set*

$$\widehat{\varepsilon}_u(x) = \widehat{e}_u(a) - \int_a^x c(y)\,e_q(y)\,dy, \tag{3.2a}$$

$$\widehat{\varepsilon}_q(x) = \widehat{e}_q(a) - \int_a^x d(y)\,e_u(y)\,dy. \tag{3.2b}$$

The we have

$$c\,\widehat{\varepsilon}_q + \widehat{\varepsilon}_u' = -c\,S_q \quad \textit{in } (0,1), \tag{3.3a}$$

$$\widehat{\varepsilon}_q' + d\,\widehat{\varepsilon}_u = -d\,S_u \quad \textit{in } (0,1), \tag{3.3b}$$

$$\widehat{\varepsilon}_u = 0 \quad \textit{on } \{0,1\}, \tag{3.3c}$$

where

$$S_u = e_u - \widehat{\varepsilon}_u, \tag{3.4a}$$

$$S_q = e_q - \widehat{\varepsilon}_q, \tag{3.4b}$$

Note that $(-c\,S_q, -d\,S_u)$ is nothing but the residual of the function $(\widehat{\varepsilon}_u, \widehat{\varepsilon}_q)$.

Proof. By the definition of the function $(\widehat{\varepsilon}_u, \widehat{\varepsilon}_q)$, (3.2), we immediately obtain that

$$\begin{aligned} c\, e_q + \widehat{\varepsilon}_u{}' &= 0 \quad \text{in } (0,1),\\ \widehat{\varepsilon}_q{}' + d\, e_u &= 0 \quad \text{in } (0,1). \end{aligned}$$

The first two equations now follow by the definition of (S_u, S_q). Let us prove the third equation. By the definition of $\widehat{\varepsilon}_u$, (3.2a), we have that

$$\widehat{\varepsilon}_u\,(0) = \widehat{e}_u(a) + \int_0^a c(y)\, e_q(y)\, dy = 0,$$

by the error equation (3.1c). Finally, by the definition of $\widehat{\varepsilon}_u$, (3.2a), we have that

$$\begin{aligned} \widehat{\varepsilon}_u\,(1) &= \widehat{e}_u(a) - \int_1^a c(y)\, e_q(y)\, dy \\ &= \widehat{e}_u(a) - \widehat{e}_u(b) - \int_b^a c(y)\, e_q(y)\, dy \end{aligned}$$

by the error equation (3.1d). Hence

$$\widehat{\varepsilon}_u\,(1) = - < \widehat{e}_u, 1\cdot n >_{\partial\mathcal{T}_h} - (c\, e_q, 1)_{\mathcal{T}_h} = \ 0,$$

by the error equation (3.1a) with $v \equiv 1$. This proves the third equation and completes the proof. ☐

Step 2: The error representation formulas. Next, we show that the auxiliary function $(\widehat{\varepsilon}_u, \widehat{\varepsilon}_q)$ can itself be expressed in terms of (S_u, S_q). To do that, we need to introduce two Green's functions associated with our model problem (1.1). To do that, we are going to use the jump of a function η at the point $x \in (0,1)$, namely,

$$[\![\varphi\, n]\!](x) := \varphi(x^-) - \varphi(x^+).$$

For any given function $d \in L^\infty(0,1)$ and any point $x \in (0,1)$, G_x is the solution of

$$-\left(\frac{1}{c}\,(G_x)'\right)' + d\, G_x = 0 \quad \text{in } (0,x)\cup(x,1), \tag{3.5a}$$

$$G_x = 0 \quad \text{on } \{0,1\}, \tag{3.5b}$$

$$[\![G_x\, n]\!] = 0 \quad \text{at } \{x\}, \tag{3.5c}$$

$$[\![\frac{1}{c}\,(G_x)'\, n]\!] = 1 \quad \text{at } \{x\}. \tag{3.5d}$$

Similarly, we denote by $\mathcal{G}_x$ the solution of

$$-\left(\frac{1}{c}\,(\mathcal{G}_x)'\right)' + d\, \mathcal{G}_x = 0 \quad \text{in } (0,x)\cup(x,1), \tag{3.6a}$$

$$\mathcal{G}_x = 0 \quad \text{on } \{0,1\}, \tag{3.6b}$$

$$[\![\mathcal{G}_x\, n]\!] = 1, \quad \text{at } \{x\}, \tag{3.6c}$$

$$[\![\frac{1}{c}\,(\mathcal{G}_x)'\, n]\!] = 0 \quad \text{at } \{x\}. \tag{3.6d}$$

LEMMA 3.2 (Error representation formulas). *For all $x \in (0,1)$, we have*

$$e_u(x) = S_u(x) + (\Gamma_{uu}(x,\cdot), S_u) + (\Gamma_{uq}(x,\cdot), S_q), \tag{3.7a}$$

$$e_q(x) = S_q(x) + (\Gamma_{qu}(x,\cdot), S_u) + (\Gamma_{qq}(x,\cdot), S_q), \tag{3.7b}$$

where, for all $y \in (0,1) \setminus \{ x \}$,

$$\Gamma_{uu}(x,y) = -\, d(y)\, G_x(y), \tag{3.8a}$$

$$\Gamma_{uq}(x,y) = -\, (G_x)'(y), \tag{3.8b}$$

$$\Gamma_{qu}(x,y) = -\, d(y)\, \mathcal{G}_x(y), \tag{3.8c}$$

$$\Gamma_{qq}(x,y) = -\, (\mathcal{G}_x)'(y). \tag{3.8d}$$

To prove this result, we are going to use the following simple auxiliary result.

PROPOSITION 3.3. *Let (q, u) be the solution of*

$$\begin{aligned} c\,\mathsf{q} + \mathsf{u}' &= c\,R_1 && \text{in } (0,1),\\ \mathsf{q}' + d\,\mathsf{u} &= R_2 && \text{in } (0,1),\\ \mathsf{u} &= 0 && \text{on } \{0,1\}. \end{aligned}$$

Then, for any $x \in (0,1)$,

$$\begin{aligned} \mathsf{u}(x) &= (R_1, (G_x)') + (R_2, G_x),\\ \mathsf{q}(x) &= (R_1, (\mathcal{G}_x)') + (R_2, \mathcal{G}_x). \end{aligned}$$

We are now ready to prove Lemma 3.2.

Proof. Since, by definition of (S_u, S_q) and $(\widehat{\varepsilon}_u\,, \widehat{\varepsilon}_q\,)$, (3.4) and (3.2), respectively, we have

$$(e_u, e_q) = (S_u, S_q) + (\widehat{\varepsilon}_u\,, \widehat{\varepsilon}_q\,).$$

Then, by Proposition 3.3 with $(R_1, R_2) = -(S_q, d\,S_u)$, we see that

$$\begin{aligned} \widehat{\varepsilon}_u\,(x) &= -\,((G_x)', S_q) - (G_x, d\,S_u) = (\Gamma_{uu}(x,\cdot), S_u) + (\Gamma_{uq}(x,\cdot), S_q),\\ \widehat{\varepsilon}_q\,(x) &= -\,((\mathcal{G}_x)', S_q) - (\mathcal{G}_x, d\,S_u) \;= (\Gamma_{qu}(x,\cdot), S_u) + (\Gamma_{qq}(x,\cdot), S_q), \end{aligned}$$

by definition of the functions $\Gamma_{uu}, \Gamma_{uq}, \Gamma_{qu}$ and Γ_{qq}. This completes the proof. □

3.2. Characterization of the function (S_q, S_u). Thanks to the error representation result, it is clear that, to obtain our error estimates, we only have to find estimates of the function (S_q, S_u). Here, we obtain a complete characterization of the function (S_q, S_u) which we then use to estimate it.

We proceed in two steps. First, we use the error equation and the definition of the numerical traces to obtain a characterization of the function (S_u, S_q) in the subdomain D. Then, we characterize the function (S_u, S_q) in the intervals $(0, a)$ and $(b, 1)$ by using the definition of the extension (q_h, u_h) therein.

Step 1: Characterization of the function (S_u, S_q) in $\mathcal{T}_h$. By using the error equations and the definition of the numerical traces, we can characterize the function (S_u, S_q) in $\mathcal{T}_h$ as we see in the following result.

Lemma 3.4. *For $j = 1, \dots, N$, we have*

$$(S_u, z)_{I_j} = 0 \quad \forall z \in \mathcal{P}_{k_j-1}(I_j), \tag{3.9a}$$
$$(S_q, z)_{I_j} = 0 \quad \forall z \in \mathcal{P}_{k_j-1}(I_j), \tag{3.9b}$$

and

$$(S_q - \tau\, S_u)(x_{j-1}^+) = 0, \tag{3.10a}$$
$$(S_q - \tau\, S_u)(x_j^-) = 0. \tag{3.10b}$$

Proof. Let us begin by proving that

$$\widehat{\varepsilon}_u = \widehat{e}_u \quad \text{on } \{x_j\}_{j=0}^N, \tag{3.11a}$$
$$\widehat{\varepsilon}_q = \widehat{e}_q \quad \text{on } \{x_j\}_{j=0}^N, \tag{3.11b}$$

Taking v to be the characteristic function of the interval I_j in the error equation (3.1a), we obtain

$$(c\, e_q, 1)_{I_j} + < \widehat{e}_u \cdot n, 1 >_{\partial I_j} = 0.$$

Since, by definition of $\widehat{\varepsilon}_u$, (3.2a), $\widehat{\varepsilon}_u{}' = -c\, e_q$, the above identity becomes

$$< (\widehat{\varepsilon}_u - \widehat{e}_u) \cdot n, 1 >_{\partial I_j} = 0,$$

and so

$$\widehat{\varepsilon}_u(x_j) - \widehat{e}_u(x_j) = \widehat{\varepsilon}_u(x_{j-1}) - \widehat{e}_u(x_{j-1}) =: \lambda.$$

Since for $j = 1$, we have that

$$\lambda = \widehat{\varepsilon}_u(a) - \widehat{e}_u(a) = 0,$$

by the definition of $\widehat{\varepsilon}_u$, (3.2a), we obtain (3.11a). The property (3.11b) can be proven in a similar way.

Now, let us prove the orthogonality properties (3.9). Taking the support of v to be the interval I_j, the error equation (3.1a) becomes

$$-(\widehat{\varepsilon}_u{}', v)_{I_j} + < \widehat{\varepsilon}_u \cdot n, v >_{\partial I_j} + (e_u, v')_{I_j} = 0,$$

by the definition of $\widehat{\varepsilon}_u$, (3.2a), and property (3.11a). Hence,

$$(-\widehat{\varepsilon}_u + e_u, v')_{I_j} = 0,$$

for all $v \in \mathcal{P}_{k_j}(I_j)$. This proves (3.9a). The property (3.9b) can be proven in a similar way.

Next, we prove the identities (3.10). By the definition of S_q, (3.4b), and property (3.11b), we have

$$\begin{aligned} S_q(x_{j-1}^+) &= e_q(x_{j-1}^+) - \widehat{e}_q(x_{j-1}) \\ &= \tau(x_{j-1}^+)\,(e_u(x_{j-1}^+) - \widehat{e}_u(x_{j-1})) \end{aligned}$$

by the definition of the numerical traces, (2.6). Hence

$$S_q(x_{j-1}^+) = \tau(x_{j-1}^+)\, S_u(x_{j-1}^+),$$

by definition of S_u, (3.4a), and property (3.11a). This proves (3.10a). The identity (3.10b) can be proved in a similar way. This completes the proof. ∎

Next, we use the information provided by this lemma, to express (S_q, S_u) as the sum of a local interpolation error and a compact operator applied to (S_q, S_u). To state the result, we need to introduce the projections $\pi_j^\pm$ defined on $H^1(I_j)$, for $j = 1, \dots, N$, by

$$(\pi_j^\pm w - w, v)_{I_j} = 0 \quad \forall v \in \mathcal{P}_{k_j-1}(I_j), \tag{3.12a}$$

$$\pi_j^+ w(x_{j-1}^+) = w(x_{j-1}^+) \quad \text{and} \quad \pi_j^- w(x_j^-) = w(x_j^-). \tag{3.12b}$$

We also need to introduce two more operators. For $j = 1, \dots, N$, we have, on the interval I_j,

$$\mathcal{I}^\pm \phi = (I - \pi_j^\pm)\, \phi, \tag{3.13a}$$

$$\mathcal{K}^\pm(\phi) = \mathcal{I}^\pm\, \mathsf{Int}_j(\phi), \tag{3.13b}$$

where $\mathsf{Int}_j(\phi)(x) := \int_{x_{j-1}}^x \phi(s)\, ds.$

We are now ready to state our result.

Corollary 3.5. *For $j = 1, \dots, N$, we have, on the interval I_j,*

$$S_u = \frac{\tau^+\, \mathcal{J}_u^+ + \tau^-\, \mathcal{J}_u^-}{\tau^+ + \tau^-} - \frac{\mathcal{J}_q^+ - \mathcal{J}_q^-}{\tau^+ + \tau^-}, \tag{3.14a}$$

$$S_q = \frac{\tau^-\, \mathcal{J}_q^+ + \tau^+\, \mathcal{J}_q^-}{\tau^+ + \tau^-} - \frac{\tau^+\, \tau^-}{\tau^+ + \tau^-}\, \left(\mathcal{J}_u^+ - \mathcal{J}_u^-\right), \tag{3.14b}$$

where, $\tau^+ = \tau(x_{j-1}^+)$ and $\tau^- = \tau(x_j^-)$. Here we have, for any $x \in I_j$,

$$\mathcal{J}_u^\pm(x) = \mathcal{I}^\pm u(x) + \mathcal{K}^\pm(c\, S_q)(x) + (\mathfrak{F}_{qu}^\pm(x,\cdot), S_u) + (\mathfrak{F}_{qq}^\pm(x,\cdot), S_q), \tag{3.15a}$$

$$\mathcal{J}_q^\pm(x) = \mathcal{I}^\pm q(x) + \mathcal{K}^\pm(d\, S_u)(x) + (\mathfrak{F}_{uu}^\pm(x,\cdot), S_u) + (\mathfrak{F}_{uq}^\pm(x,\cdot), S_q), \tag{3.15b}$$

where

$$\mathfrak{F}_{\cdot\cdot}^\pm(x,y) = \mathcal{K}^\pm(\mathfrak{G}_{\cdot\cdot}(\cdot,y))(x), \tag{3.16}$$

and

$$\mathfrak{G}_{uu}(x,y) = d(x)\, \Gamma_{uu}(x,y), \tag{3.17a}$$

$$\mathfrak{G}_{uq}(x,y) = d(x)\, \Gamma_{uq}(x,y), \tag{3.17b}$$

$$\mathfrak{G}_{qu}(x,y) = c(x)\, \Gamma_{qu}(x,y)), \tag{3.17c}$$

$$\mathfrak{G}_{qq}(x,y) = c(x)\, \Gamma_{qq}(x,y). \tag{3.17d}$$

Proof. We begin by noting that we can write

$$S_q = \frac{\tau^-\, A + \tau_+\, B}{\tau^+ + \tau^-} \quad \text{and} \quad S_u = \frac{-A + B}{\tau^+ + \tau^-},$$

where $A := S_q - \tau^+ S_u$ and $B := S_q + \tau^- S_u$. Since by Lemma 3.4 and the definition of the projections $\pi_j^\pm$ (3.12), we have that, on I_j, $\pi_j^+ A = 0$ and $\pi_j^- B = 0$, we have that

$$
\begin{aligned}
A &= (I - \pi_j^+)A = \mathcal{J}^+ q + \mathcal{K}^+(d\, e_u) - \tau^+ \left(\mathcal{J}^+ u + \mathcal{K}^+(c\, e_q)\right), \\
B &= (I - \pi_j^-)B = \mathcal{J}^- q + \mathcal{K}^-(d\, e_u) + \tau^- \left(\mathcal{J}^- u + \mathcal{K}^-(c\, e_q)\right).
\end{aligned}
$$

The result now follows by noting that, by the error representation formulas of Lemma 3.2, for any $x \in I_j$,

$$
\begin{aligned}
\mathcal{K}^\pm(d\, e_u)(x) &= \mathcal{K}^\pm(d\, S_u)(x) + (\mathfrak{F}_{uu}^\pm(x,\cdot), S_u) + (\mathfrak{F}_{uq}^\pm(x,\cdot), S_q), \\
\mathcal{K}^\pm(c\, e_q)(x) &= \mathcal{K}^\pm(c\, S_q)(x) + (\mathfrak{F}_{qu}^\pm(x,\cdot), S_u) + (\mathfrak{F}_{qq}^\pm(x,\cdot), S_q).
\end{aligned}
$$

This completes the proof. □

Step 2: Characterization of the function (S_u, S_q) **outside** $\mathcal{T}_h$. Now we come to a crucial part of the analysis of the new method we are proposing here. Indeed, here we incorporate into the analysis the mechanism by which the conditions on $\partial\Omega$ are *transferred* into those on $\partial\mathsf{D}$.

LEMMA 3.6. *We have*

$$
S_u(x) = 0 \qquad \text{for } x \in I_1^{ext} \cup I_N^{ext}, \tag{3.18a}
$$

$$
S_q(x) = \begin{cases} \sigma_1(x) + \int_a^x d(s)\, S_u(s)\, ds & \text{for } x \in \overset{\star}{I}_1, \\ \sigma_N(x) - \int_b^x d(s)\, S_u(s)\, ds & \text{for } x \in \overset{\star}{I}_N, \end{cases} \tag{3.18b}
$$

where

$$
\sigma_1(x) = e_q(x) - \widehat{e}_q(a) - \int_a^x d(s) \int_0^\zeta c(\zeta)\, e_q(\zeta)\, d\zeta\, ds \qquad \text{for } x \in \overset{\star}{I}_1, \tag{3.19a}
$$

$$
\sigma_N(x) = e_q(x) - \widehat{e}_q(a) + \int_b^x d(s) \int_\zeta^1 c(\zeta)\, e_q(\zeta)\, d\zeta\, ds \qquad \text{for } x \in \overset{\star}{I}_N. \tag{3.19b}
$$

Proof. The identity (3.18a) is a direct consequence of the definition of $\widehat{\varepsilon}_u$, (3.2a), and that of u_h outside $\mathcal{T}_h$, (2.2b).

Let us prove the identities (3.18b). By definition of S_q, (3.4), we have that, for $x \in \overset{\star}{I}_1$,

$$
\begin{aligned}
S_q(x) &= e_q(x) - \widehat{e}_q(a) + \int_a^x d(s)\, e_u(s)\, ds, \\
&= e_q(x) - \widehat{e}_q(a) + \int_a^x d(s)\, (\widehat{\varepsilon}_u + S_u)(s)\, ds,
\end{aligned}
$$

by definition of S_u, (3.4). Hence

$$
S_q(x) = \sigma_1(x) + \int_a^x d(s)\, S_u(s)\, ds,
$$

by definition of $\widehat{\varepsilon}_u$, (3.2a) and (3.1c), and that of σ_1. The proof of the expression for $x \in \overset{\star}{I}_N$ is similar. This completes the proof. □

3.3. Estimates of the function (S_u, S_q). It only remains to obtain suitable estimates of the function (S_u, S_q). They are gathered in the following key result.

LEMMA 3.7. *If the condition* (2.13) *is satisfied then, for* $j = 1, \dots, N$,

$$\| S_u \|_{L^\infty(I_j)} \leq \Theta_{u,j}\ \kappa_j,$$
$$\| S_q \|_{L^\infty(I_j)} \leq \Theta_{q,j}\ \kappa_j,$$

and, for $\nu = 1$ *and* $\nu = N$,

$$\| S_u \|_{L^\infty(I_\nu^{ext})} = 0,$$
$$\| S_q \|_{L^\infty(I_\nu^{ext})} \leq \Theta_{q,\nu}^{ext}\ \kappa_{\rho,\nu}^{ext},$$

where

$$\Theta_{u,j} = C\, |\, u\, |_{W^{k_j+1,\infty}(I_j)} + C\, \kappa\, \|\!| (u, q) \|\!|,$$
$$\Theta_{q,j} = C\, |\, q\, |_{W^{k_j+1,\infty}(I_j)} + C\, \kappa\, \|\!| (u, q) \|\!|,$$
$$\Theta_{q,\nu}^{ext} = C\, |\, q\, |_{W^{k_\nu+1,\infty}(\overset{\star}{I}_\nu)} + C\, \kappa\, \|\!| (u, q) \|\!|,$$

and

$$\kappa = \max_{1 \leq j \leq N} \{\underline{\kappa}_j, \kappa_{\rho,j}\} + \max_{\nu=1,N} \kappa_{\rho,\nu}^{ext},$$

for some constant C independent of the discretization parameters.

The proof of these estimated is presented at the end of this Section.

3.4. Proofs of the main theorems. Next, we use the results obtained in the previous subsections to prove our main results.

Proof of the local estimates in the uniform norm. We are now ready to prove Theorem 2.1. Indeed, by the error representation formulas of Lemma 3.2 and the orthogonality properties (3.9) of Lemma 3.4, we have

$$\begin{aligned} e_u(x) = S_u(x) &+ \sum_{\ell=1}^{N} (\Gamma_{uu}(x,\cdot) - p_{uu,\ell}, S_u)_{I_\ell} + \sum_{\ell=1}^{N} (\Gamma_{uq}(x,\cdot) - p_{uq,\ell}, S_q)_{I_\ell} \\ &+ \sum_{\nu=1,N} (\Gamma_{uq}(x,\cdot), S_q)_{I_\nu^{ext}}, \\ e_q(x) = S_q(x) &+ \sum_{\ell=1}^{N} (\Gamma_{qu}(x,\cdot) - p_{qu,\ell}, S_u)_{I_\ell} + \sum_{\ell=1}^{N} (\Gamma_{qq}(x,\cdot) - p_{qq,\ell}, S_q)_{I_\ell} \\ &+ \sum_{\nu=1,N} (\Gamma_{qq}(x,\cdot), S_q)_{I_\nu^{ext}}, \end{aligned}$$

for any polynomials $p_{\cdot\cdot,\ell}$ in $\mathcal{P}_{k_\ell - 1}(I_\ell)$. The result now follows by using the estimates of (S_u, S_q) Lemma 3.7 and the approximation result of Proposition 6.4 in the Appendix.

Let us illustrate this procedure by estimating $\| e_u \|_{L^\infty(I_j)}$. We have

$$\begin{aligned}\| e_u \|_{L^\infty(I_j)} \leq \| S_u \|_{L^\infty(I_j)} &+ \| \Gamma_{uu} \|_{L^\infty(I_j,I_j)} \| S_u \|_{L^\infty(I_j)} h_j \\ &+ \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \| \frac{\partial^{k_\ell}}{\partial y^{k_\ell}} \Gamma_{uu} \|_{L^\infty(I_j,I_\ell)} \| S_u \|_{L^\infty(I_\ell)} \mathsf{K}(h_\ell; k_\ell, k_\ell)\, h_\ell \\ &+ \| \Gamma_{uq} \|_{L^\infty(I_j,I_j)} \| S_q \|_{L^\infty(I_j)} h_j \\ &+ \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \| \frac{\partial^{k_\ell}}{\partial y^{k_\ell}} \Gamma_{qu} \|_{L^\infty(I_j,I_\ell)} \| S_q \|_{L^\infty(I_\ell)} \mathsf{K}(h_\ell; k_\ell, k_\ell)\, h_\ell \\ &+ \sum_{\nu=1,N} \| \Gamma_{uq} \|_{L^\infty(I_j,I_\nu^{ext})} \| S_q \|_{L^\infty(I_\nu^{ext})} h_\nu^{ext}.\end{aligned}$$

Inserting the estimates of Lemma 3.7, we get

$$\begin{aligned}\| e_u \|_{L^\infty(I_j)} \leq (\Theta_{u,j} &+ (\| \Gamma_{uu} \|_{L^\infty(I_j\times I_j)} \Theta_{u,j} + \| \Gamma_{uq} \|_{L^\infty(I_j\times I_j)} \Theta_{q,j})\, h_j)\, \kappa_j \\ &+ \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \| \frac{\partial^{k_\ell}}{\partial y^{k_\ell}} \Gamma_{uu} \|_{L^\infty(I_j\times I_\ell)} \Theta_{u,\ell}\, \mathsf{K}(h_\ell; k_\ell, k_\ell)\, \kappa_\ell\, h_\ell \\ &+ \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \| \frac{\partial^{k_\ell}}{\partial y^{k_\ell}} \Gamma_{qu} \|_{L^\infty(I_j\times I_\ell)} \Theta_{q,\ell}\, \mathsf{K}(h_\ell; k_\ell, k_\ell)\, \kappa_\ell\, h_\ell \\ &+ \sum_{\nu=1,N} \| \Gamma_{uq} \|_{L^\infty(I_j\times I_\nu^{ext})} \Theta_{q,\nu}^{ext}\, \kappa_{\rho,\nu}^{ext}\, h_\nu^{ext},\end{aligned}$$

and the estimate follows. This completes the proof of Theorem 2.1.

Proof of the estimates of the numerical traces. Since, by (3.2), we have that, on all nodes,

$$(\widehat{e}_u, \widehat{e}_q) = (\widehat{\varepsilon}_u, \widehat{\varepsilon}_q) = (e_u - S_u, e_q - S_q),$$

by definition of (S_u, S_q), (3.4), we obtain, by the error representation formulas of Lemma 3.2,

$$\begin{aligned}\widehat{e}_u(x_j) &= (\Gamma_{uu}(x_j,\cdot), S_u) + (\Gamma_{uq}(x_j,\cdot), S_q),\\ \widehat{e}_q(x_j) &= (\Gamma_{qu}(x_j,\cdot), S_u) + (\Gamma_{qq}(x_j,\cdot), S_q).\end{aligned}$$

By the orthogonality properties (3.9) of Lemma 3.4, we get

$$\begin{aligned}\widehat{e}_u(x_j) = &\sum_{\ell=1}^{N} (\Gamma_{uu}(x_j,\cdot) - p_{uu,\ell}, S_u)_{I_\ell} + \sum_{\ell=1}^{N} (\Gamma_{uq}(x_j,\cdot) - p_{uq,\ell}, S_q)_{I_\ell} \\ &+ \sum_{\nu=1,N} (\Gamma_{uq}(x_j,\cdot), S_q)_{I_\nu^{ext}},\end{aligned}$$

and

$$\widehat{e}_q(x_j) = \sum_{\ell=1}^{N}(\Gamma_{qu}(x_j,\cdot) - p_{uu,\ell}, S_u)_{I_\ell} + \sum_{\ell=1}^{N}(\Gamma_{qq}(x_j,\cdot) - p_{uq,\ell}, S_q)_{I_\ell}$$
$$+ \sum_{\nu=1,N}(\Gamma_{qq}(x_j,\cdot), S_q)_{I_\nu^{ext}},$$

for any polynomials $p_{\cdot\cdot,\ell}$ in $\mathcal{P}_{k_\ell-1}(I_\ell)$. The result now follows exactly as in the previous two proofs. This completes the proof of Theorem 2.2.

Proof of the estimates in negative-order norms. Note that if φ_u and φ_q are smooth functions, then, by the error representation formulas of Lemma 3.2, we have

$$(e_u, \varphi_u) = (\varphi_u + (\Gamma_{uu}, \varphi_u), S_u) + ((\varphi_u, \Gamma_{uq}), S_q),$$
$$(e_q, \varphi_q) = (\varphi_q + (\Gamma_{qu}, \varphi_q), S_u) + ((\varphi_q, \Gamma_{qq}), S_q),$$

where $(\Gamma_{\cdot\cdot}, \varphi_\cdot)(y) = \int_0^1 \Gamma_{\cdot\cdot}(x,y)\, \varphi_\cdot(x)\, dx$. Now, by the orthogonality properties (3.9) of Lemma 3.4, we get

$$(e_u, \varphi_u) = \sum_{\ell=1}^{N}(\varphi_u + (\Gamma_{uu}, \varphi_u) - p_{uu,\ell}, S_u)_{I_\ell}$$
$$+ \sum_{\ell=1}^{N}((\varphi_u, \Gamma_{uq}) - p_{uq,\ell}, S_q)_{I_\ell} + \sum_{\nu=1,N}((\varphi_u, \Gamma_{uq}), S_q)_{I_\nu^{ext}},$$

and

$$(e_q, \varphi_q) = \sum_{\ell=1}^{N}(\varphi_q + (\Gamma_{qu}, \varphi_u) - p_{qu,\ell}, S_u)_{I_\ell}$$
$$+ \sum_{\ell=1}^{N}((\varphi_q, \Gamma_{qq}) - p_{qq,\ell}, S_q)_{I_\ell} + \sum_{\nu=1,N}((\varphi_q, \Gamma_{qq}), S_q)_{I_\nu^{ext}},$$

for any polynomials $p_{\cdot\cdot,\ell}$ in $\mathcal{P}_{k_\ell-1}(I_\ell)$. The result now follows by using the estimates of (S_u, S_q) Lemma 3.7 and the approximation result of Proposition 6.4 in the Appendix. This completes the proof of Theorem 2.3.

Let us note that when $d = 0$ and c is a constant, we have that $\Gamma_{uu} = \Gamma_{qu} = 0$ and $\Gamma_{uq}(x_j,\cdot)$ and $\Gamma_{qq}(x_j,\cdot)$ are constant on each element; see their definition (3.8). Hence, if $k_j \geq 1$ for $j = 1,\ldots,N$ and $a = 0, b = 1$, we readily see that $\widehat{e}_u(x_j) = \widehat{e}_q(x_j) = 0$ for all $j = 1,\ldots,N$, and that $e_u = S_u, e_q = S_q$ on Ω. We are going to see that in this case, S_u, S_q can be expressed as local approxiamtion errors, as claimed in Section 2.

4. Proof of the estimates of the function (S_u, S_q). It remains to prove the estimates of (S_u, S_q) given in Lemma 3.7. This proof constitutes the main novelty of this paper, as far as the error analysis is concerned. Since the proof is rather involved, we divide it in several distinct steps.

Step 1: An approximation result. We begin by stating estimates of the operators $\mathcal{I}^{\pm}$ and $\mathcal{K}^{\pm}$ defined by (3.13), that is, by obtaining approximation properties of the projections $\pi_j^{\pm}$ defined in (3.12).

The estimates we seek can be readily deduced from estimates of

$$\mathcal{I}_{n,I}^{\pm} f = f - \pi_{n,I}^{\pm} f,$$

on an arbitrary interval $I = (\alpha, \beta)$, where $\pi_{n,I}^{\pm} f$ is the element of $\mathcal{P}_n(I)$ determined by the conditions

$$\begin{aligned}(\pi_{n,I}^{\pm} f - f, v)_I = &\ 0 \quad \forall v \in \mathcal{P}_{n-1}(I),\\ \pi_{n,}^{+} f(\alpha) = &\ f(\alpha) \quad \text{and} \quad \pi_{n,I}^{-} f(\beta) = \ f(\beta).\end{aligned}$$

The estimates we are interested in are contained in the following result; see the Appendix for a proof.

Proposition 4.1. *If $f \in W^{k+1,\infty}(I)$, we have that*

$$\max\{\| \mathcal{I}_{n,I}^{\pm} f \|_{L^\infty(I)}, \| \mathcal{J}_{n,I}^{+} f - \mathcal{J}_{n,I}^{-} f \|_{L^\infty(I)} \} \le \mathsf{C}\, \mathsf{K}(\frac{\beta-\alpha}{2}; n+1, k+1) \| f^{(k+1)} \|_{L^\infty(I)}$$

provided $n \ge k$, where C is independent of α, β, f, n and k. Here, $\mathsf{K}(\cdot;\cdot,\cdot)$ is the function defined by (2.11).

Step 2: Preliminary estimates of (S_u, S_q) in $\mathcal{T}_h$. We begin by obtaining a preliminary estimate of the quantities

(4.1a) $\quad \mathsf{S}_{u,j} := \| S_u \|_{L^\infty(I_j)} \quad$ and $\quad \mathsf{S}_{q,j} := \| S_q \|_{L^\infty(I_j)} \qquad$ for $j = 1, \dots, N$.

We are also going to set,

(4.1b) $\quad \mathsf{S}_{q,\nu}^{ext} := \| S_q \|_{L^\infty(I_\nu^{ext})} \qquad$ for $\nu = 1$ and $\nu = N$.

The first estimate follows directly from Corollary 3.5 and the approximation properties of the projections $\pi_j^{\pm}$; see Proposition 4.1.

Lemma 4.2. *For $j = 1, \dots, N$, we have*

$$\mathsf{S}_{u,j} \le \mathsf{I}_{u,j} + \sum_{\ell=1}^{N} \mathsf{G}_{uu,j\ell}\, \mathsf{S}_{u,\ell} + \sum_{\ell=1}^{N} \mathsf{G}_{uq,j\ell}\, \mathsf{S}_{q,\ell} + \sum_{\nu=1,N} \mathsf{G}_{uq,j\nu}^{ext}\, \mathsf{S}_{q,\nu}^{ext},$$

$$\mathsf{S}_{q,j} \le \mathsf{I}_{q,j} + \sum_{\ell=1}^{N} \mathsf{G}_{qu,j\ell}\, \mathsf{S}_{u,\ell} + \sum_{\ell=1}^{N} \mathsf{G}_{qq,j\ell}\, \mathsf{S}_{q,\ell} + \sum_{\nu=1,N} \mathsf{G}_{qq,j\nu}^{ext}\, \mathsf{S}_{q,\nu}^{ext},$$

where

$$\begin{aligned}\mathsf{I}_{u,j} := &\ \kappa_j\, I_{u,j}, \quad I_{u,j} := \mathsf{C} \max\{1,\theta\} \,|\, u \,|_{W^{k_j+1,\infty}(I_j)},\\ \mathsf{I}_{q,j} := &\ \kappa_j\, I_{q,j}, \quad I_{q,j} := \mathsf{C} \max\{1,\eta\} \,|\, q \,|_{W^{k_j+1,\infty}(I_j)},\end{aligned}$$

and

$$\begin{aligned}\mathsf{G}_{\cdot\cdot,jj} := &\ \underline{\kappa}_j\, G_{\cdot\cdot,jj}, \quad G_{\cdot\cdot,jj} := \mathsf{C}\, (\alpha_{\cdot\cdot,j} + h_j\, g_{\cdot\cdot,jj}),\\ \mathsf{G}_{\cdot\cdot,j\ell} := &\ \kappa_j\, G_{\cdot\cdot,j\ell}, \quad G_{\cdot\cdot,j\ell} := \mathsf{C}\, h_\ell\, g_{\cdot\cdot,j\ell} \quad \textit{if } j \ne \ell,\\ \mathsf{G}_{\cdot\cdot,j\nu}^{ext} := &\ \kappa_j\, G_{\cdot\cdot,j\nu}^{ext}, \quad G_{\cdot\cdot,j\nu}^{ext} := \mathsf{C}\, h_\nu^{ext}\, g_{\cdot\cdot,j\nu}^{ext}.\end{aligned}$$

Here

$$\alpha_{uu,j} := \theta \, \| \, d \, \|_{L^\infty(I_j)}, \qquad \alpha_{uq,j} := \| \, c \, \|_{L^\infty(I_j)},$$
$$\alpha_{qu,j} := \| \, d \, \|_{L^\infty(I_j)}, \qquad \alpha_{qq,j} := \eta \, \| \, c \, \|_{L^\infty(I_j)},$$

and

$$g_{u\cdot,j\ell} := \left\| \frac{\partial^{(1-\delta_{j\ell})\,k_j}}{\partial x^{(1-\delta_{j\ell})\,k_j}} \mathfrak{G}_{q\cdot} \right\|_{L^\infty(I_j\times I_\ell)} + \theta \left\| \frac{\partial^{(1-\delta_{j\ell})\,k_j}}{\partial x^{(1-\delta_{j\ell})\,k_j}} \mathfrak{G}_{u\cdot} \right\|_{L^\infty(I_j\times I_\ell)},$$
$$g_{q\cdot,j\ell} := \left\| \frac{\partial^{(1-\delta_{j\ell})\,k_j}}{\partial x^{(1-\delta_{j\ell})\,k_j}} \mathfrak{G}_{u\cdot} \right\|_{L^\infty(I_j\times I_\ell)} + \eta \left\| \frac{\partial^{(1-\delta_{j\ell})\,k_j}}{\partial x^{(1-\delta_{j\ell})\,k_j}} \mathfrak{G}_{q\cdot} \right\|_{L^\infty(I_j\times I_\ell)},$$

and

$$g^{ext}_{u\cdot,j\nu} := \left\| \frac{\partial^{k_j}}{\partial x^{k_j}} \mathfrak{G}_{q\cdot} \right\|_{L^\infty(I_j\times I^{ext}_\nu)} + \theta \left\| \frac{\partial^{k_j}}{\partial x^{k_j}} \mathfrak{G}_{u\cdot} \right\|_{L^\infty(I_j\times I^{ext}_\nu)},$$
$$g^{ext}_{q\cdot,j\nu} := \left\| \frac{\partial^{k_j}}{\partial x^{k_j}} \mathfrak{G}_{u\cdot} \right\|_{L^\infty(I_j\times I^{ext}_\nu)} + \eta \left\| \frac{\partial^{k_j}}{\partial x^{k_j}} \mathfrak{G}_{q\cdot} \right\|_{L^\infty(I_j\times I^{ext}_\nu)}.$$

Note that when $d = 0$, c is a constant, $k_j \geq 1$ for $j = 1, \ldots, N$ and $a = 0, b = 1$, we have that $\mathsf{S}_{u,j} \leq \mathsf{I}_{u,j}$ and $\mathsf{S}_{q,j} \leq \mathsf{I}_{q,j}$ for $j = 1, \ldots, N$. Thus, S_u and S_q are bounded by local approximation errors, as claimed at the end of Section 3.

Proof. By Corollary 3.5, we have that, for $j = 1, \ldots, N$,

$$\mathsf{S}_{u,j} \leq \max\{\| \, \mathfrak{J}^+_u \, \|_{L^\infty(I_j)}, \| \, \mathfrak{J}^-_u \, \|_{L^\infty(I_j)} \} + \theta \, \| \, \mathfrak{J}^+_q - \mathfrak{J}^-_q \, \|_{L^\infty(I_j)},$$
$$\mathsf{S}_{q,j} \leq \max\{\| \, \mathfrak{J}^+_q \, \|_{L^\infty(I_j)}, \| \, \mathfrak{J}^-_q \, \|_{L^\infty(I_j)} \} + \eta \, \| \, \mathfrak{J}^+_u - \mathfrak{J}^-_u \, \|_{L^\infty(I_j)},$$

and that, by the definition of $\mathfrak{J}^\pm_u$ and $\mathfrak{J}^\pm_q$,(3.15),

$$\begin{aligned}
\| \, \mathfrak{J}^\pm_u \, \|_{L^\infty(I_j)} \leq{}& \| \, \mathcal{I}^\pm u \, \|_{L^\infty(I_j)} + \| \, \mathcal{K}^\pm(c\, S_q) \, \|_{L^\infty(I_j)} \\
&+ \sum_{\ell=1}^{N} h_\ell \, \left(\| \, \mathfrak{F}^\pm_{qu} \, \|_{L^\infty(I_j\times I_\ell)} \, \mathsf{S}_{u,\ell} + \| \, \mathfrak{F}^\pm_{qq} \, \|_{L^\infty(I_j\times I_\ell)} \, \mathsf{S}_{q,\ell} \right), \\
&+ \sum_{\nu=1,N} h^{ext}_\nu \, \| \, \mathfrak{F}^\pm_{qq} \, \|_{L^\infty(I_j\times I^{ext}_\nu)} \, \mathsf{S}^{ext}_{q,\nu}, \\
\| \, \mathfrak{J}^\pm_q \, \|_{L^\infty(I_j)} \leq{}& \| \, \mathcal{I}^\pm q \, \|_{L^\infty(I_j)} + \| \, \mathcal{K}^\pm(d\, S_u) \, \|_{L^\infty(I_j)} \\
&+ \sum_{\ell=1}^{N} h_\ell \, \left(\| \, \mathfrak{F}^\pm_{uu} \, \|_{L^\infty(I_j\times I_\ell)} \, \mathsf{S}_{u,\ell} + \| \, \mathfrak{F}^\pm_{uq} \, \|_{L^\infty(I_j\times I_\ell)} \, \mathsf{S}_{q,\ell} \right) \\
&+ \sum_{\nu=1,N} h^{ext}_\nu \, \| \, \mathfrak{F}^\pm_{uq} \, \|_{L^\infty(I_j\times I^{ext}_\nu)} \, \mathsf{S}^{ext}_{q,\nu}.
\end{aligned}$$

By Proposition 4.1, we also have

$$\| \, \mathcal{I}^\pm u \, \|_{L^\infty(I_j)} \leq \mathsf{C} \, \kappa_j \, | \, u \, |_{W^{k_j+1,\infty}(I_j)},$$
$$\| \, \mathcal{I}^\pm q \, \|_{L^\infty(I_j)} \leq \mathsf{C} \, \kappa_j \, | \, q \, |_{W^{k_j+1,\infty}(I_j)},$$
$$\| \, \mathcal{K}^\pm(c\, S_q) \, \|_{L^\infty(I_j)} \leq \mathsf{C} \, \underline{\kappa}_j \, \| \, c \, \|_{L^\infty(I_j)} \, \mathsf{S}_{q,j},$$
$$\| \, \mathcal{K}^\pm(d\, S_u) \, \|_{L^\infty(I_j)} \leq \mathsf{C} \, \underline{\kappa}_j \, \| \, d \, \|_{L^\infty(I_j)} \, \mathsf{S}_{u,j},$$

and

$$\| \mathfrak{F}^{\pm}_{\cdot\cdot} \|_{L^\infty(I_j\times I_\ell)} \le \mathfrak{C}_{j\ell} \, \| \frac{\partial^{(1-\delta_{j\ell})\,k_j}}{\partial x^{(1-\delta_{j\ell})\,k_j}} \mathfrak{G}_{\cdot\cdot} \|_{L^\infty(I_j\times I_\ell)},$$

$$\| \mathfrak{F}^{\pm}_{\cdot q} \|_{L^\infty(I_j\times I_\nu^{ext})} \le \mathfrak{C}^{ext}_{j\nu} \, \| \frac{\partial^{k_j}}{\partial x^{k_j}} \mathfrak{G}_{\cdot q} \|_{L^\infty(I_j\times I_\nu^{ext})},$$

where $\mathfrak{C}_{j\ell} := \mathsf{C}\,\left(\delta_{j\ell}\,\underline{\kappa}_j + (1-\delta_{j\ell})\,\kappa_j\right)$ and $\mathfrak{C}^{ext}_{j\nu} := \mathsf{C}\;\kappa_j$. The result now follows by simple algebraic manipulations. This completes the proof. □

Step 3: Relating S_q outside $\mathcal{T}_h$ with (S_q, S_u) inside $\mathcal{T}_h$. Our objective here is to obtain bounds of S_q on $(0,a)$ and $(b,1)$ in terms of bounds of (S_q, S_u) inside $\mathcal{T}_h$. To do that, we use the Lemma 3.6 and some estimates relating the uniform norms of the same polynomial in *different* intervals. We begin with the following result.

Lemma 4.3. *For $\nu = 1$ and $\nu = N$, we have that, for any integer m_ν,*

$$\| S_q \|_{L^\infty(I_\nu^{ext})} \le \; 2\,\rho_\nu^{m_\nu} \inf_{p\in\mathcal{P}_{m_\nu}(\overset{\star}{I}_\nu)} \| \sigma_\nu - p \|_{L^\infty(\overset{\star}{I}_\nu)} + \rho_\nu^{m_\nu} \, \| \sigma_\nu \|_{L^\infty(I_\nu)},$$

where ρ_ν is defined by (2.12e).

Proof. We prove the result for $\nu = 1$; the case $\nu = N$ can be proven in a similar way. Since $S_q = \sigma_1$ on the interval I_1^{ext}, see (3.18), we have

$$\| S_q \|_{L^\infty(I_1^{ext})} \le \; \| \sigma_1 - p \|_{L^\infty(I_1^{ext})} + \| p \|_{L^\infty(I_1^{ext})},$$

for any polynomial $p \in \mathcal{P}_{m_1}(\overset{\star}{I}_1)$. By Theorem 1.10 in Rivlin [9], we get that

$$\| S_q \|_{L^\infty(I_1^{ext})} \le \; \| \sigma_1 - p \|_{L^\infty(I_1^{ext})} + T_{m_1}(1 + 2\,h_1^{ext}/h_1) \, \| p \|_{L^\infty(I_1)},$$

where T_ℓ is the Chebyshev polynomial of degree ℓ. Since $T_{m_1}(1 + 2\,h_1^{ext}/h_1) \ge 1$, we obtain

$$\| S_q \|_{L^\infty(I_1^{ext})} \le \; T_{m_1}(1 + 2\,h_1^{ext}/h_1) \, \Big(\| \sigma_1 - p \|_{L^\infty(I_1^{ext})} + \| p \|_{L^\infty(I_1)} \Big),$$

and since $T_{m_1}(1 + 2\,h_1^{ext}/h_1) \le \rho_1^{m_1}$,

$$\begin{aligned} \| S_q \|_{L^\infty(I_1^{ext})} &\le \rho_1^{m_1} \, \Big(\| \sigma_1 - p \|_{L^\infty(I_1^{ext})} + \| p \|_{L^\infty(I_1)} \Big) \\ &\le \rho_1^{m_1} \, \Big(\| \sigma_1 - p \|_{L^\infty(I_1^{ext})} + \| \sigma_1 - p \|_{L^\infty(I_1)} + \| \sigma_1 \|_{L^\infty(I_1)} \Big) \\ &\le \rho_1^{m_1} \, \Big(2 \, \| \sigma_1 - p \|_{L^\infty(\overset{\star}{I}_1)} + \| \sigma_1 \|_{L^\infty(I_1)} \Big) . \end{aligned}$$

The result now follows from the fact that the polynomial p was arbitrarily chosen. This completes the proof. □

Next, we estimate the approximation of the functions σ_ν for $\nu = 1$ and $\nu = N$.

Lemma 4.4. *For $\nu = 1$ and $\nu = N$, there is a polynomial $p_\nu \in \mathcal{P}_{3\,k_\nu}(\overset{\star}{I}_\nu)$ such that*

$$\| \sigma_\nu - p_\nu \|_{L^\infty(\overset{\star}{I}_\nu)} \le \; \mathsf{I}^{ext}_{q,\nu} + \mathsf{I}^{ext}_{e_q,\nu},$$

where

$$\mathsf{I}_{q,\nu}^{ext} := \kappa_\nu^{ext}\, I_{q,\nu}^{ext}, \qquad I_{q,\nu}^{ext} := C_{q,\nu}^{ext}\, |\, q\,|_{W^{k_\nu+1,\infty}(\overset{\star}{I}_\nu)},$$
$$\mathsf{I}_{e_q,\nu}^{ext} := \kappa_\nu^{ext}\, I_{e_q,\nu}^{ext}, \quad I_{e_q,\nu}^{ext} := C_{e_q,\nu}^{ext}\, \|\, e_q\,\|_{L^\infty(\overset{\star}{I}_\nu)}.$$

Here

$$C_{q,\nu}^{ext} := \mathsf{C} + \mathsf{C}\,(h_\nu\,(1+r_\nu))^2 \cdot \left(\|\, d\,\|_{L^\infty(\overset{\star}{I}_\nu)} + \left(\frac{\mathsf{c}\, h_\nu\,(1+r_\nu)}{k_\nu+1}\right)^{k_\nu+1} |\, d\,|_{W^{k_\nu,\infty}(\overset{\star}{I}_\nu)} \right) \cdot \left(\|\, c\,\|_{L^\infty(\overset{\star}{I}_\nu)} + \left(\frac{\mathsf{c}\, h_\nu\,(1+r_\nu)}{k_\nu+1}\right)^{k_\nu+1} |\, c\,|_{W^{k_\nu,\infty}(\overset{\star}{I}_\nu)} \right),$$
$$C_{e_q,\nu}^{ext} := \left(\|\, d\,\|_{L^\infty(\overset{\star}{I}_\nu)} |\, c\,|_{W^{k_\nu,\infty}(\overset{\star}{I}_\nu)} + \|\, c\,\|_{L^\infty(\overset{\star}{I}_\nu)} |\, d\,|_{W^{k_\nu,\infty}(\overset{\star}{I}_\nu)} + |\, d\,|_{W^{k_\nu,\infty}(\overset{\star}{I}_\nu)} |\, c\,|_{W^{k_\nu,\infty}(\overset{\star}{I}_\nu)} \right) \left(\frac{\mathsf{c}\, h_\nu\,(1+r_\nu)}{k_\nu+1}\right).$$

Proof. Let us prove the result for the case $\nu = 1$. To construct the polynomial $p_1 \in \mathcal{P}_{3\,k_1}(\overset{\star}{I}_1)$ under consideration, we begin by noting that, by definition of σ_1, (3.19a), we can write

$$\sigma_1(x) = T_1(x) + T_2(x) + T_3(x),$$

where

$$T_1(x) = e_q(x) - \widehat{e}_q(a) - \int_x^a p_d(s) \int_0^s p_c(\zeta)\, e_q(\zeta)\, d\zeta\, ds,$$
$$T_2(x) = -\int_x^a p_d(s) \int_0^s (c - p_c)(\zeta)\, e_q(\zeta)\, d\zeta\, ds,$$
$$T_3(x) = -\int_x^a (d - p_d)(s) \int_0^s c(\zeta)\, e_q(\zeta)\, d\zeta\, ds,$$

for arbitrary polynomials p_d and p_c of degree $k_1 - 1$. The form of the term T_1 suggests to consider functions of the form

$$p(x) = (p_q - q_h)(x) - \widehat{e}_q(a) - \int_x^a p_d(s) \int_0^s p_c(\zeta)\,(p_q - q_h)(\zeta)\, d\zeta\, ds,$$

where p_q is an arbitrary polynomial of degree k_1, as candidates for the polynomial p_1 we are seeking; note that p is a polynomial of degree $3\,k_1$. Indeed, we take p_1 to be exactly p when p_q, p_d and p_c are the best uniform approximations to q, d and c, respectively. Hence

$$\sigma_1(x) - p_1(x) = T_1(x) - p(x) + T_2(x) + T_3(x),$$

where

$$T_1(x) - p_1(x) = (q - p_q)(x) - \int_x^a p_d(s) \int_0^s p_c(\zeta)\,(q - p_q)(\zeta)\, d\zeta\, ds.$$

and the result is obtained from the following elementary estimates

$$
\begin{aligned}
\| T_1 - p_1 \|_{L^\infty(\overset{\star}{I}_1)} &\leq (1 + (h_1 + h_1^{ext})^2 \, \| p_d \|_{L^\infty(\overset{\star}{I}_1)} \, \| p_c \|_{L^\infty(\overset{\star}{I}_1)}) \, \| q - p_q \|_{L^\infty(\overset{\star}{I}_1)}, \\
\| T_2 \|_{L^\infty(\overset{\star}{I}_1)} &\leq (h_1 + h_1^{ext})^2 \, \| p_d \|_{L^\infty(\overset{\star}{I}_1)} \, \| c - p_c \|_{L^\infty(\overset{\star}{I}_1)} \, \| e_q \|_{L^\infty(\overset{\star}{I}_1)} \\
\| T_3 \|_{L^\infty(\overset{\star}{I}_1)} &\leq (h_1 + h_1^{ext})^2 \, \| d - p_d \|_{L^\infty(\overset{\star}{I}_1)} \, \| c \|_{L^\infty(\overset{\star}{I}_1)} \, \| e_q \|_{L^\infty(\overset{\star}{I}_1)},
\end{aligned}
$$

and the estimates for best uniform approximations in the Appendix. The proof for $\nu = N$ is similar. This completes the proof. ▯

The following result is a straightforward consequence of the two previous lemmas and the formula (3.18b) giving S_q on I^ext_ν.

Corollary 4.5. *For $\nu = 1$ and $\nu = N$, we have*

$$
\mathsf{S}_{q,\nu}^{ext} \leq \; \rho_\nu^{3\,k_\nu} \, (\mathsf{I}_{q,\nu}^{ext} + \mathsf{I}_{e_q,\nu}^{ext} + \mathsf{S}_{q,\nu} + h_\nu \, \| d \|_{L^\infty(I_\nu)} \, \mathsf{S}_{u,\nu}).
$$

Step 4: Eliminating $\mathsf{S}_{q,\nu}^{ext}$ from the estimates of $\mathsf{S}_{q,j}$ and $\mathsf{S}_{u,j}$. We can now eliminate the quantities $\mathsf{S}_{q,\nu}^{ext}$, for $\nu = 1$ and $\nu = N$, from the estimates of $\mathsf{S}_{q,j}$ and $\mathsf{S}_{u,j}$ for $j = 1, \dots, N$ obtained in Lemma 4.2 by a straightforward use of Corollary 4.5. However, Corollary 4.5 suggests to estimate, not $\mathsf{S}_{q,j}$ and $\mathsf{S}_{u,j}$, but the quantities

$$
\mathbb{S}_{\cdot,j} := \rho_j^{3\,k_j} \, \mathsf{S}_{\cdot,j}, \tag{4.2}
$$

where $\rho_j = 1$ for $j = 2, \dots, N-1$ and ρ_1 and ρ_n are defined at the end of subsection 2.3.

Lemma 4.6. *For $j = 1, \dots, N$, we have*

$$
\begin{aligned}
\mathbb{S}_{u,j} &\leq \mathbb{I}_{u,j} + \sum_{\ell=1}^{N} \big(\mathbb{G}_{uu,j\ell} \, \mathbb{S}_{u,\ell} + \mathbb{G}_{uq,j\ell} \, \mathbb{S}_{q,\ell}\big), \\
\mathbb{S}_{q,j} &\leq \mathbb{I}_{q,j} + \sum_{\ell=1}^{N} \big(\mathbb{G}_{qu,j\ell} \, \mathbb{S}_{u,\ell} + \mathbb{G}_{qq,j\ell} \, \mathbb{S}_{q,\ell}\big).
\end{aligned}
$$

Here

$$
\begin{aligned}
\mathbb{I}_{u,j} &:= \kappa_{\rho,j} \, \Big(I_{u,j} + \sum_{\nu=1,N} G_{uq,j\nu}^{ext} \, \big(\kappa_{\rho,\nu}^{ext} \, I_{q,\nu}^{ext} + \kappa_{\rho,\nu}^{ext} \, I_{e_q,\nu}^{ext}\big)\Big), \\
\mathbb{I}_{q,j} &:= \kappa_{\rho,j} \, \Big(I_{q,j} + \sum_{\nu=1,N} G_{qq,j\nu}^{ext} \, \big(\kappa_{\rho,\nu}^{ext} \, I_{q,\nu}^{ext} + \kappa_{\rho,\nu}^{ext} \, I_{e_q,\nu}^{ext}\big)\Big).
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\mathbb{G}_{\cdot u,jj} &:= \underline{\kappa}_j \, G_{\cdot u,jj} + \kappa_{\rho,j} \sum_{\nu=1,N} G_{\cdot q,j\nu}^{ext} \, h_\nu \, \| d \|_{L^\infty(I_\nu)} \, \delta_{j\nu}, \\
\mathbb{G}_{\cdot q,jj} &:= \underline{\kappa}_j \, G_{\cdot q,jj} + \kappa_{\rho,j} \sum_{\nu=1,N} G_{\cdot q,j\nu}^{ext} \, \delta_{\ell\nu},
\end{aligned}
$$

and, for $j \neq \ell$*,*

$$\mathbb{G}_{\cdot u,j\ell} := \kappa_{\rho,j} \left(G_{\cdot u,j\ell}\, \rho_\ell^{-3\,k_\ell} + \sum_{\nu=1,N} G^{ext}_{\cdot q,j\nu}\, h_\nu \,\|\, d \,\|_{L^\infty(I_\nu)}\, \delta_{\ell\nu} \right),$$
$$\mathbb{G}_{\cdot q,j\ell} := \kappa_{\rho,j} \left(G_{\cdot q,j\ell}\, \rho_\ell^{-3\,k_\ell} + \sum_{\nu=1,N} G^{ext}_{\cdot q,j\nu}\, \delta_{\ell\nu} \right).$$

The next step is to use this result to estimate the quantities $\mathbb{S}_{u,\cdot}$ and $\mathbb{S}_{q,\cdot}$ solely in terms of the quantities $\mathbb{I}_{u,\cdot}$ and $\mathbb{I}_{q,\cdot}$. To be able to do that, we must ensure that the coefficients $\mathbb{G}_{\cdot\cdot,\cdot\cdot}$ are *small* enough. We are going to postpone this task for the next step where we show that we can make those coefficients as small as we want by simply decreasing the step size h_j or by raising the polynomial degree k_j.

Let us now to obtain an upper bound of $(\mathbb{S}_{u,j}, \mathbb{S}_{q,j})$. It will be expressed in terms of the quantities $\mathbb{S}_u := \max_{1\le j\le N} \mathbb{S}_{u,j}$ and $\mathbb{S}_q := \max_{1\le j\le N} \mathbb{S}_{q,j}$.

Lemma 4.7. *Assume that*

$$\mathbb{G}_{\cdot\cdot,jj} \le \frac{\epsilon_j}{(1+2\,\epsilon_j)}, \tag{4.3}$$

is satisfied for positive parameter ϵ_j*. Then, for* $j = 1, \dots, N$*, we have*

$$\mathbb{S}_{u,j} \le \mathfrak{I}_{u,j} + \mathfrak{S}_{uu,j}\, \mathbb{S}_u + \mathfrak{S}_{uq,j}\, \mathbb{S}_q,$$
$$\mathbb{S}_{q,j} \le \mathfrak{I}_{q,j} + \mathfrak{S}_{qu,j}\, \mathbb{S}_u + \mathfrak{S}_{qq,j}\, \mathbb{S}_q,$$

where

$$\mathfrak{I}_{u,j} := \mathbb{I}_{u,j} + \epsilon_j \left(\mathbb{I}_{u,j} + \mathbb{I}_{q,j}\right),$$
$$\mathfrak{S}_{uu,j} := \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \mathbb{G}_{uu,j\ell} + \epsilon_j \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} (\mathbb{G}_{uu,j\ell} + \mathbb{G}_{qu,j\ell}),$$
$$\mathfrak{S}_{uq,j} := \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \mathbb{G}_{uq,j\ell} + \epsilon_j \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} (\mathbb{G}_{uq,j\ell} + \mathbb{G}_{qq,j\ell}),$$

and

$$\mathfrak{I}_{q,j} := \mathbb{I}_{q,j} + \epsilon_j \left(\mathbb{I}_{u,j} + \mathbb{I}_{q,j}\right),$$
$$\mathfrak{S}_{qq,j} := \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \mathbb{G}_{qu,j\ell} + \epsilon_j \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} (\mathbb{G}_{qu,j\ell} + \mathbb{G}_{uu,j\ell}),$$
$$\mathfrak{S}_{qq,j} := \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \mathbb{G}_{qq,j\ell} + \epsilon_j \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} (\mathbb{G}_{qq,j\ell} + \mathbb{G}_{uq,j\ell}).$$

Proof. From the previous lemma, we have that

$$\mathbb{S}_{u,j} \le \mathbb{J}_{u,j} + \mathbb{G}_{uu,jj}\, \mathbb{S}_{u,j} + \mathbb{G}_{uq,jj}\, \mathbb{S}_{q,j},$$
$$\mathbb{S}_{q,j} \le \mathbb{J}_{q,j} + \mathbb{G}_{qu,jj}\, \mathbb{S}_{u,j} + \mathbb{G}_{qq,jj}\, \mathbb{S}_{q,j},$$

where

$$\mathbb{J}_{u,j} = \mathbb{I}_{u,j} + \sum_{\substack{\ell=1 \\ \ell\neq j}}^{N} \mathbb{G}_{uu,j\ell}\, \mathbb{S}_u + \sum_{\substack{\ell=1 \\ \ell\neq j}}^{N} \mathbb{G}_{uq,j\ell}\, \mathbb{S}_q,$$

$$\mathbb{J}_{q,j} = \mathbb{I}_{q,j} + \sum_{\substack{\ell=1 \\ \ell\neq j}}^{N} \mathbb{G}_{qu,j\ell}\, \mathbb{S}_u + \sum_{\substack{\ell=1 \\ \ell\neq j}}^{N} \mathbb{G}_{qq,j\ell}\, \mathbb{S}_q,$$

By the condition (4.3), we get that

$$\mathbb{S}_{u,j} \leq \mathbb{J}_{u,j} + \frac{\epsilon_j}{(1+2\,\epsilon_j)}\, \mathbb{S}_{u,j} + \frac{\epsilon_j}{(1+2\,\epsilon_j)}\, \mathbb{S}_{q,j},$$
$$\mathbb{S}_{q,j} \leq \mathbb{J}_{q,j} + \frac{\epsilon_j}{(1+2\,\epsilon_j)}\, \mathbb{S}_{u,j} + \frac{\epsilon_j}{(1+2\,\epsilon_j)}\, \mathbb{S}_{q,j},$$

and hence that

$$\mathbb{S}_{u,j} \leq \mathbb{J}_{u,j} + \epsilon_j\, \left(\mathbb{J}_{u,j} + \mathbb{J}_{q,j}\right),$$
$$\mathbb{S}_{q,j} \leq \mathbb{J}_{u,j} + \epsilon_j\, \left(\mathbb{J}_{u,j} + \mathbb{J}_{q,j}\right).$$

The result now follows by inserting the definition of $\mathbb{J}_{\cdot,j}$ and by using the fact that $\mathbb{S}_u := \max_{1\leq j\leq N} \mathbb{S}_{u,j}$ and that $\mathbb{S}_q : \max_{1\leq j\leq N} \mathbb{S}_{q,j}$. This completes the proof. ◻

Finally, we obtain estimates of $(\mathbb{S}_u, \mathbb{S}_u)$.

LEMMA 4.8. *Assume that the smallness condition*

$$\max_{1\leq j\leq N} \sum_{\substack{\ell=1 \\ \ell\neq j}}^{N} \{\, \mathbb{G}_{\cdot\cdot,j\ell}\,\} \leq \frac{\delta}{(1+2\,\epsilon)\,(1+2\,\epsilon+2\,\delta)}, \tag{4.4}$$

holds for some positive parameters δ *and* $\epsilon := \max_{1\leq j\leq N} \epsilon_j$. *Then*

$$\mathbb{S}_u \leq \mathbb{I}_u + (\epsilon+\delta)\, \left(\mathbb{I}_u + \mathbb{I}_q\right),$$
$$\mathbb{S}_q \leq \mathbb{I}_q + (\epsilon+\delta)\, \left(\mathbb{I}_u + \mathbb{I}_q\right),$$

where $\mathbb{I}_u = \max_{1\leq j\leq N} \mathbb{I}_{u,j}$ *and* $\mathbb{I}_q = \max_{1\leq j\leq N} \mathbb{I}_{q,j}$.

Proof. Setting $\mathbb{T}_u := \max_{1\leq j\leq N} \mathcal{J}_{u,j}$ and $\mathbb{T}_q := \max_{1\leq j\leq N} \mathcal{J}_{q,j}$, we get, by the previous lemma and the assumption (4.4), that

$$\mathbb{S}_u \leq \mathbb{T}_u + \gamma\, \mathbb{S}_u + \gamma\, \mathbb{S}_q,$$
$$\mathbb{S}_q \leq \mathbb{T}_q + \gamma\, \mathbb{S}_u + \gamma\, \mathbb{S}_q,$$

where $\gamma = \frac{\delta\,(1+2\,\epsilon)}{(1+2\,\delta)(1+2\,\epsilon+2\,\delta)}$. This immediately implies that

$$\mathbb{S}_u \leq \mathbb{T}_u + \frac{\gamma}{1-2\,\gamma}\, (\mathbb{T}_u + \mathbb{T}_q),$$
$$\mathbb{S}_q \leq \mathbb{T}_q + \frac{\gamma}{1-2\,\gamma}\, (\mathbb{T}_u + \mathbb{T}_q).$$

The result now follows by inserting the definitions of $\mathbb{T}_u, \mathbb{T}_q$, those of $\mathcal{J}_{u,j}, \mathcal{J}_{q,j}$ given in Lemma 4.7, and by rewriting γ in terms of the parameters ϵ and δ. This completes the proof. ◻

We end this step by inserting the estimates in this last lemma into those of Lemma 4.7 and expressing the result in terms of $(\mathsf{S}_{u,j}, \mathsf{S}_{q,j})$ by using (4.2); we obtain the following result.

Corollary 4.9. *Assume that the smallness conditions* (4.3) *and* (4.4) *are satisfied. Then, for* $j = 1, \ldots, N$*, we have*

$$\begin{aligned} \mathsf{S}_{u,j} &\le \rho_j^{-3\,k_j}\, \mathbb{S}_{u,j}, \\ \mathsf{S}_{q,j} &\le \rho_j^{-3\,k_j}\, \mathbb{S}_{q,j}, \end{aligned}$$

and for $\nu = 1$ *and* $\nu = N$*,*

$$\mathsf{S}_{q,\nu}^{ext} \le \kappa_{\rho,\nu}^{ext}\, I_{q,\nu}^{ext} + \kappa_{\rho,\nu}^{ext}\, I_{e_q,\nu}^{ext} + \mathbb{S}_{q,\nu} + h_\nu \,\|\, d\,\|_{L^\infty(I_\nu)} \mathbb{S}_{u,\nu},$$

where, for $j = 1, \ldots, N$*,*

$$\begin{aligned} \mathbb{S}_{u,j} \le \mathbb{I}_{u,j} + \epsilon_j\,(\mathbb{I}_{u,j} + \mathbb{I}_{q,j}) &+ \mathfrak{S}_{uu,j}\;\;(\mathbb{I}_u + (\epsilon+\delta)\,(\mathbb{I}_u + \mathbb{I}_q)) \\ &+ \mathfrak{S}_{uq,j}\;\;(\mathbb{I}_q + (\epsilon+\delta)\,(\mathbb{I}_u + \mathbb{I}_q))\,, \\ \mathbb{S}_{q,j} \le \mathbb{I}_{q,j} + \epsilon_j\,(\mathbb{I}_{u,j} + \mathbb{I}_{q,j}) &+ \mathfrak{S}_{qu,j}\;\;(\mathbb{I}_u + (\epsilon+\delta)\,(\mathbb{I}_u + \mathbb{I}_q)) \\ &+ \mathfrak{S}_{qq,j}\;\;(\mathbb{I}_u + (\epsilon+\delta)\,(\mathbb{I}_u + \mathbb{I}_q))\,. \end{aligned}$$

Step 5: The smallness conditions. Here, we show that the smallness conditions (4.3) and (4.4) can always be satisfied. We begin by estimating the quantities $\mathbb{G}_{\cdot\cdot,\cdot\cdot}$ involved in those conditions.

Proposition 4.10. *We have*

$$\begin{aligned} \mathbb{G}_{\cdot\cdot,jj} &\le \vartheta_{jj}\;\max\{\underline{\kappa}_j, \kappa_{\rho,j}\}, \\ \mathbb{G}_{\cdot\cdot,j\ell} &\le \vartheta_{j\ell}\;\kappa_{\rho,j}\, h_\ell \qquad \textit{if} \qquad j \ne \ell, \end{aligned}$$

where

$$\begin{aligned} \vartheta_{jj} &:= \gamma_{jj} + \gamma_{jj}^{ext} \sum_{\nu=1,N} h_\nu^{ext}\, \delta_{j\nu}, \\ \gamma_{jj} &:= \mathsf{C}\,(\alpha_{\cdot,\cdot,j} + h_j\, g_{\cdot\cdot,jj}), \\ \gamma_{jj}^{ext} &:= \mathsf{C} \max_{\nu=1,N}\; g_{\cdot\cdot,j\nu}^{ext}\;\max\{1, h_\nu \,\|\, d\,\|_{L^\infty(I_\nu)}\}\,\delta_{j\nu}, \end{aligned}$$

and

$$\begin{aligned} \vartheta_{j\ell} &:= \gamma_{j\ell} + \gamma_{j\ell}^{ext} \sum_{\nu=1,N} h_\nu^{ext}\, \delta_{\ell\nu}, \\ \gamma_{j\ell} &:= \mathsf{C}\; g_{\cdot\cdot,j\ell}, \\ \gamma_{j\ell}^{ext} &:= \mathsf{C}\; g_{\cdot\cdot,j\nu}^{ext}\;\max\{1, h_\nu \,\|\, d\,\|_{L^\infty(I_\nu)}\}, \end{aligned}$$

if $\ell \ne j$*.*

Proof. Let us prove the first estimate. By using the definition of $\mathbb{G}_{\cdot\cdot,j\ell}$ in Lemma (4.6), we see that

$$\mathbb{G}_{\cdot\cdot,jj} \le \underline{\kappa}_j\; G_{\cdot\cdot,jj} + \kappa_{\rho,j} \sum_{\nu=1,N} G_{\cdot\cdot,j\nu}^{ext}\;\max\{1, h_\nu \,\|\, d\,\|_{L^\infty(I_\nu)}\}\,\delta_{j\nu}.$$

If we now use the definition of $G_{\cdot\cdot,jj}$ and $G^{ext}_{\cdot\cdot,j\nu}$ given in Lemma (4.2), we obtain

$$\begin{aligned}\mathbb{G}_{\cdot\cdot,jj} \leq \mathsf{C}\,\underline{\kappa}_j\,(\alpha_{\cdot,\cdot,j} + h_j\, g_{\cdot\cdot,jj}) \\ + \mathsf{C}\,\kappa_{\rho,j} \sum_{\nu=1,N} h^{ext}_\nu\, g^{ext}_{\cdot\cdot,j\nu}\, \max\{1, h_\nu\, \|\, d\,\|_{L^\infty(I_\nu)}\}\, \delta_{j\nu},\end{aligned}$$

and the first estimate follows.

Let us prove the second estimate. By using the definition of $\mathbb{G}_{\cdot\cdot,j\ell}$ in Lemma (4.6), we see that

$$\mathbb{G}_{\cdot\cdot,j\ell} \leq \kappa_{\rho,j}\, G_{\cdot\cdot,j\ell}\, \rho_\ell^{-3\,k_\ell} + \kappa_{\rho,j} \sum_{\nu=1,N} G^{ext}_{\cdot\cdot,j\nu}\, \max\{1, h_\nu\, \|\, d\,\|_{L^\infty(I_\nu)}\}\, \delta_{\ell\nu}.$$

If we now use the definition of $G_{\cdot\cdot,j\ell}$ and $G^{ext}_{\cdot\cdot,j\nu}$ given in Lemma (4.2), we obtain

$$\begin{aligned}\mathbb{G}_{\cdot\cdot,j\ell} \leq \mathsf{C}\,\kappa_{\rho,j}\, h_\ell\, g_{\cdot\cdot,j\ell}\, \rho_\ell^{-3\,k_\ell} \\ + \mathsf{C}\,\kappa_{\rho,j}\, h_\ell \sum_{\nu=1,N} h^{ext}_\nu\, g^{ext}_{\cdot\cdot,j\nu}\, \max\{1, h_\nu\, \|\, d\,\|_{L^\infty(I_\nu)}\}\, \delta_{\ell\nu},\end{aligned}$$

and the second estimate follows. This completes the proof. ☐

We now give a simple sufficient condition for the smallness conditions (4.3) and (4.4) to hold. It is written in terms of quantities we define next. Set

(4.5a) $$A := \vartheta_A \max_{1\leq j\leq N} \{\underline{\kappa}_j, \kappa_{\rho,j}\}, \qquad \vartheta_A \geq \max_{1\leq j\leq N} \vartheta_{jj},$$

(4.5b) $$B := \vartheta_B \max_{1\leq j\leq N} \kappa_{\rho,j}, \qquad \vartheta_B \geq \max_{1\leq j\leq N} \max_{\substack{1\leq \ell\leq N\\ \ell\neq j}} \vartheta_{j\ell}.$$

Note that ϑ_A and ϑ_B can be chosen as to be independent of the numerical method under consideration; see the definition of $\vartheta_{\cdot\cdot}$ in Lemma 4.10.

Lemma 4.11. *Assume that*

(4.6) $$A \leq A_0 \quad \textit{and} \quad B \leq B_0,$$

where A_0 and B_0 are positive real numbers such that $A_0 < \frac{1}{2}$ and $B_0 < \frac{1}{2} - A_0$. Then, the smallness conditions (4.3) *and* (4.4) *hold with*

(4.7) $$\epsilon := c_{A_0}\, A \quad \textit{and} \quad \delta := c_{B_0}\, B,$$

where $c_{A_0} := 1/(1 - 2\,A_0)$ and $c_{B_0} := 1/((1 - 2\,A_0)\,(1 - 2\,A_0 - 2\,B_0))$. Note that a simple computation gives us that, under the assumptions of this result,

(4.8a) $$\max_{1\leq j\leq N} \{\, \mathbb{G}_{\cdot\cdot,jj}\,\} \leq A$$

(4.8b) $$\max_{1\leq j\leq N} \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \{\, \mathbb{G}_{\cdot\cdot,j\ell}\,\} \leq B.$$

Proof. The inequalities (4.8) are a direct consequence of Lemma 4.10. To prove that the smallness conditions (4.3) and (4.4) hold, we only have to show that $A \leq \epsilon/(1 + 2\,\epsilon)$ and that $B \leq \delta/((1 + 2\,\epsilon)\,(1 + 2\,\epsilon + 2\,\delta))$, respectively.

Let us prove the first inequality. First, we note that since $A_0 \in (0,1/2)$, the constant c_{A_0} is well defined and is positive. Then we have,

$$\begin{aligned}
\frac{\epsilon}{(1+2\,\epsilon)} &= \frac{c_{A_0}\,A}{(1+2\,c_{A_0}\,A)} && \text{by our choice of } \epsilon,\\
&\geq \frac{c_{A_0}\,A}{(1+2\,c_{A_0}\,A_0)} && \text{since } A \leq A_0 \text{ and } c_{A_0} > 0,\\
&= A,
\end{aligned}$$

and the inequality follows.

The second inequality can be obtained in a similar way. This completes the proof.
$\square$

Step 6: Refining the estimates of $(\mathsf{S}_{u,j}, \mathsf{S}_{q,j})$. Next, we further refine the estimates obtained in the previous result in order to clearly see its different components. We begin by estimating $(\mathbb{S}_{u,j}, \mathbb{S}_{q,j})$.

LEMMA 4.12. *Assume that the conditions* (4.6) *are satisfied. Then we have, for* $j = 1, \ldots, N$,

$$\begin{aligned}
\mathbb{S}_{u,j} &\leq \mathbb{S}_{u,j}(u,q) + \mathbb{S}_{u,j}(e_q),\\
\mathbb{S}_{q,j} &\leq \mathbb{S}_{q,j}(u,q) + \mathbb{S}_{q,j}(e_q),
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{S}_{\cdot,j}(u,q) &= \kappa_{\rho,j}\,(\,I_{\cdot,j} \;+\; HOT_{\cdot,j}(u,q)\,),\\
\mathbb{S}_{\cdot,j}(e_q) &= \kappa_{\rho,j}\,C^{ext}\,\max_{\nu=1,N}\kappa^{ext}_{\rho,\nu}\,\max_{\nu=1,N} I^{ext}_{e_q,\nu}.
\end{aligned}$$

Here, the high-order term is given by

$$HOT_{\cdot,j}(u,q) := HOT^{loc}_{\cdot,j}(u,q) + HOT^{glb}_{\cdot,j}(u,q) + HOT^{ext}_{\cdot,j}(u,q),$$

where

$$\begin{aligned}
HOT^{loc}_{\cdot,j}(u,q) &:= \epsilon_j\,(I_{u,j} + I_{q,j}),\\
HOT^{glb}_{\cdot,j}(u,q) &:= C^{glb}\,\max_{1\leq j\leq N}\kappa_{\rho,j}\,\max_{1\leq j\leq N}\{I_{u,j}, I_{q,j}\},\\
HOT^{ext}_{\cdot,j}(u,q) &:= C^{ext}\,\max_{\nu=1,N}\kappa^{ext}_{\rho,\nu}\,\max_{\nu=1,N} I^{ext}_{q,\nu},
\end{aligned}$$

and

$$\begin{aligned}
C^{glb} &:= 2\,\vartheta_B\,(1+2\epsilon)(1+2\epsilon+2\delta),\\
C^{ext} &:= (1+2\epsilon + C^{glb}\max_{1\leq j\leq N}\kappa_{\rho,j})\,G^{ext},\\
G^{ext} &:= \max\{\max_{1\leq j\leq N}\sum_{\nu=1,N} G^{ext}_{uq,j\nu},\, \max_{1\leq j\leq N}\sum_{\nu=1,N} G^{ext}_{qq,j\nu}\}.
\end{aligned}$$

Here ϵ *and* δ *are given by* (4.7).

Proof. By Corollary 4.9, we have that

$$\mathbb{S}_{\cdot,j} \leq \mathbb{I}_{\cdot,j} + \epsilon\,(\mathbb{I}_{u,j} + \mathbb{I}_{q,j}) + \mathfrak{S}_{\cdot u,j}\ \ (\mathbb{I}_u + (\epsilon + \delta)\,(\mathbb{I}_u + \mathbb{I}_q)) \\ + \mathfrak{S}_{\cdot q,j}\ \ (\mathbb{I}_q + (\epsilon + \delta)\,(\mathbb{I}_u + \mathbb{I}_q))\,.$$

Since, by the definition of $\mathfrak{S}_{\cdot\cdot,j}$ in Lemma 4.7, we have that

$$\mathfrak{S}_{\cdot\cdot,j} \leq (1 + 2\,\epsilon)\ \max_{\xi,\zeta=u,q} \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \mathbb{G}_{\xi\zeta,j\ell},$$

and since, by the Lemma 4.11,

$$\sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \mathbb{G}_{\cdot\cdot,j\ell} \leq \vartheta_B\,\kappa_{\rho,j},$$

we readily obtain that

$$\mathbb{S}_{\cdot,j} \leq \mathbb{I}_{\cdot,j} + \epsilon_j\,(\mathbb{I}_{u,j} + \mathbb{I}_{q,j}) + \vartheta_B\,\kappa_{\rho,j}\ (1 + 2\,\epsilon)\,(1 + 2\,\epsilon + 2\,\delta)\,(\mathbb{I}_u + \mathbb{I}_q) \\ \leq \mathbb{I}_{\cdot,j} + \epsilon_j\,(\mathbb{I}_{u,j} + \mathbb{I}_{q,j}) + \kappa_{\rho,j}\,C^{glb} \max_{1\leq j\leq N}\{\mathbb{I}_{u,j}, \mathbb{I}_{q,j}\}.$$

Next, we note that, by the definition of $\mathbb{I}_{\cdot,j}$ in Lemma 4.6, we have that

$$\mathbb{I}_{\cdot,j} \leq \kappa_{\rho,j}\,(I_{\cdot,j} + G^{ext}\,(\max_{\nu=1,N} I^{ext}_{q,\nu} + \max_{\nu=1,N} I^{ext}_{e_q,\nu})\ \max_{\nu=1,N} \kappa^{ext}_{\nu}\,).$$

Hence, inserting these estimates in the right-hand side of the estimate for $(\mathbb{S}_{\cdot,j}$, we obtain

$$\mathbb{S}_{\cdot,j} \leq\ \kappa_{\rho,j}\ (I_{\cdot,j} + \epsilon_j\,(I_{u,j} + I_{q,j}) + C^{glb} \max_{1\leq j\leq N}\{I_{u,j}, I_{q,j}\}) \\ +\ \kappa_{\rho,j}\ C^{ext}\,(\max_{\nu=1,N} I^{ext}_{q,\nu} + \max_{\nu=1,N} I^{ext}_{e_q,\nu})\ \max_{\nu=1,N} \kappa^{ext}_{\nu},$$

and the results follows after simple algebraic manipulations. This completes the proof. ∎

If we now use the estimates of the previous result in the inequalities of Corollary (4.9), we readily obtain the following result.

Corollary 4.13. *Assume that the conditions* (4.6) *are satisfied. Then, for* $j = 1, \ldots, N$, *we have*

$$\mathsf{S}_{u,j} \leq \mathsf{S}_{u,j}(u,q) + \mathsf{S}_{u,j}(e_q), \\ \mathsf{S}_{q,j} \leq \mathsf{S}_{q,j}(u,q) + \mathsf{S}_{q,j}(e_q),$$

where

$$\mathsf{S}_{\cdot,j}(u,q) = \kappa_j\,(\,I_{\cdot,j}\ +\ HOT_{\cdot,j}(u,q)\,), \\ \mathsf{S}_{\cdot,j}(e_q) = C^{ext}\ \kappa_j\ \max_{\nu=1,N} \kappa^{ext}_{\rho,\nu}\ \max_{\nu=1,N} I^{ext}_{e_q,\nu}.$$

Moreover, for $\nu = 1$ *and* $\nu = N$, *we have*

$$\mathsf{S}^{ext}_{q,\nu} \leq \mathsf{S}^{ext}_{q,j\nu}(u,q) + \mathsf{S}^{ext}_{q,\nu}(e_q),$$

where

$$\mathsf{S}^{ext}_{q,\nu}(u,q) := \kappa^{ext}_{\rho,\nu}\,\left(I^{ext}_{q,\nu} + I_{q,\nu} + h_\nu \,\|\, d \,\|_{L^\infty(I_\nu)}\, I_{u,\nu} + HOT^{ext}_{q,\nu}(u,q)\right),$$
$$\mathsf{S}^{ext}_{q,\nu}(e_q) := \kappa^{ext}_{\rho,\nu}\,(I^{ext}_{e_q,\nu} + HOT^{ext}_{q,\nu}),$$

and

$$HOT^{ext}_{q,\nu}(u,q) := HOT_{q,\nu}(u,q) + h_\nu \,\|\, d \,\|_{L^\infty(I_\nu)}\, HOT_{u,\nu}(u,q),$$
$$HOT^{ext}_{q,\nu}(e_q) := C^{ext}\,(1 + h_\nu \,\|\, d \,\|_{L^\infty(I_\nu)}) \max_{\nu'=1,N} \kappa^{ext}_{\rho,\nu'} \max_{\nu'=1,N} I^{ext}_{e_q,\nu'}.$$

Step 7: Estimating the error $\|\, e_q \,\|_{L^\infty(\mathsf{D}^*_h)}$. Note that, by definition of $I^{ext}_{e_q,\nu}$ in Lemma 4.4, if the function d is zero in $\mathsf{D}^*_h = \overset{\star}{I}_1 \cup \overset{\star}{I}_N$, we have that $I^{ext}_{e_q,\nu} = 0$ and the estimates of $(\mathsf{S}_u, \mathsf{S}_q)$ in Lemma 3.7 would immediately follow from our last Corollary. In the case we consider, however, since d is not necessarily zero in D^*_h, we need to carry out an additional step and obtain an estimate of $I^{ext}_{e_q,\nu}$ for $\nu = 1$ and $\nu = N$. Next, we combine the estimates obtained in the previous result with the error representation formula (3.7b) to obtain estimate $\|\, e_q \,\|_{L^\infty(\mathsf{D}_h)}$ and then deduce such a result.

Lemma 4.14. *Assume that the condition* (4.6) *holds. Assume also that*

$$\zeta_{e_q} \max_{\nu'=1,N} \kappa^{ext}_{\rho,\nu'} \leq \frac{1}{2}, \tag{4.9}$$

where

$$\zeta_{e_q} := ((\zeta_u + \zeta_q)\, C^{ext} \max_{1\leq j\leq N} \kappa_j + \zeta^{ext}_q\, C^{ext}_{e_q}) \max_{\nu=1,N} C^{ext}_{e_q,\nu}.$$
$$\zeta_u := \|\, \Gamma_{qu} \,\|_{L^\infty(\mathsf{D}^*_h \times \mathsf{D}_h)},$$
$$\zeta_q := 1 + \|\, \Gamma_{qq} \,\|_{L^\infty(\mathsf{D}^*_h \times \mathsf{D}_h)},$$
$$\zeta^{ext}_q := 1 + \sum_{\nu=1,N} h^{ext}_\nu \,\|\, \Gamma_{qq} \,\|_{L^\infty(\mathsf{D}^*_h \times I^{ext}_\nu)},$$
$$C^{ext}_{e_q} := 1 + C^{ext}\,(1 + \max_{\nu=1,N} h_\nu \,\|\, d \,\|_{L^\infty(I_\nu)}) \max_{\nu=1,N} \kappa^{ext}_{\rho,\nu}.$$

Then

$$\|\, e_q \,\|_{L^\infty(\mathsf{D}^*_h)} \leq 2\,\zeta_u \max_{1\leq j\leq N} \mathsf{S}_{u,j}(u,q) + 2\,\zeta_q \max_{1\leq j\leq N} \mathsf{S}_{q,j}(u,q) + 2\,\zeta^{ext}_q \max_{\nu=1,N} \mathsf{S}^{ext}_{q,\nu}(u,q).$$

Proof. By the error representation formula (3.7b), we have that

$$\|\, e_q \,\|_{L^\infty(\mathsf{D}^*_h)} \leq \zeta_u \max_{1\leq j\leq N} \mathsf{S}_{u,j} + \zeta_q \max_{1\leq j\leq N} \mathsf{S}_{q,j} + \zeta^{ext}_q \max_{\nu=1,N} \mathsf{S}^{ext}_{q,\nu}.$$

By Corollary (4.13), we obtain

$$\|\, e_q \,\|_{L^\infty(\mathsf{D}^*_h)} \leq \zeta_u \max_{1\leq j\leq N} \mathsf{S}_{u,j}(u,q) + \zeta_q \max_{1\leq j\leq N} \mathsf{S}_{q,j}(u,q) + \zeta^{ext}_q \max_{\nu=1,N} \mathsf{S}^{ext}_{q,\nu}(u,q)$$
$$+ ((\zeta_u + \zeta_q)\, C^{ext} \max_{1\leq j\leq N} \kappa_j + \zeta^{ext}_q\, C^{ext}_{e_q}) \max_{\nu'=1,N} \kappa^{ext}_{\rho,\nu'} \max_{\nu'=1,N} I^{ext}_{e_q,\nu'},$$

and, by the definition of $I^{ext}_{e_q,\nu'}$ in Lemma 4.4,

$$\| e_q \|_{L^\infty(\mathsf{D}_h^*)} \le \zeta_u \max_{1\le j\le N} \mathsf{S}_{u,j}(u,q) + \zeta_q \max_{1\le j\le N} \mathsf{S}_{q,j}(u,q) + \zeta_q^{ext} \max_{\nu=1,N} \mathsf{S}^{ext}_{q,\nu}(u,q)$$
$$+ \zeta_{e_q} \max_{\nu'=1,N} \kappa^{ext}_{\rho,\nu'} \| e_q \|_{L^\infty(\mathsf{D}_h^*)}.$$

and the result follows if condition (4.9) is satisfied. This completes the proof. □

Step 8: Conclusion of the proof. The last step is to insert the upper bound of $\| e_q \|_{L^\infty(\mathsf{D}_h^*)}$ obtained in the last lemma into the estimates of Corollary 4.13. We obtain the following result.

$$\mathsf{S}_{u,j} \le \kappa_j \left(S_{u,j}(u,q) + HOT^{glb} \right),$$
$$\mathsf{S}_{q,j} \le \kappa_j \left(S_{q,j}(u,q) + HOT^{glb} \right),$$

where

$$S_{\cdot,j}(u,q) := (\, I_{\cdot,j} \, + \, HOT_{\cdot,j}(u,q) \,),$$
$$HOT^{glb} := C^{ext} \max_{\nu=1,N} \kappa^{ext}_{\rho,\nu} \left(\max_{\nu=1,N} C^{ext}_{e_q,\nu} \, Z^{glb} \right).$$

Moreover, for $\nu = 1$ and $\nu = N$, we have

$$\mathsf{S}^{ext}_{q,\nu} \le \kappa^{ext}_{\rho,\nu} \left(S^{ext}_{q,j\nu}(u,q) + HOT^{glb}_* \right),$$

where

$$S^{ext}_{q,\nu}(u,q) := \left(I^{ext}_{q,\nu} + I_{q,\nu} + h_\nu \| d \|_{L^\infty(I_\nu)} I_{u,\nu} + HOT^{ext}_{q,\nu}(u,q) \right),$$
$$HOT^{glb}_* := C^{ext}_{e_q} \left(\max_{\nu=1,N} C^{ext}_{e_q,\nu} \, Z^{glb} \right),$$

and

$$Z^{glb} \le 2\,\zeta_u \max_{1\le j\le N} \kappa_j \, S_{u,j}(u,q) + 2\,\zeta_q \max_{1\le j\le N} \kappa_j \, S_{q,j}(u,q)$$
$$+ 2\,\zeta_q^{ext} \max_{\nu=1,N} \kappa^{ext}_{\rho,\nu} \, S^{ext}_{q,\nu}(u,q).$$

The estimates of Lemma 3.7 follow immediately from those of the above result. This completes the proof of Lemma 3.7.

5. Numerical results. Here, we present numerical experiments devised to validate each of our three main theoretical results, namely, Theorems 2.1, 2.3 and 2.2.

Thus, we consider the problem 1.1 with $c = 1$, $d = 1$, and take f and $u_D(0)$ in such a way that $u(x) = \sin(x)$ is the exact solution. We solve it by using the HDG method with $\tau \equiv 1$ and present its history of convergence of its h-version for several choices of $\mathsf{D} = (a,b)$ and of the polynomial degrees k_j, $j = 1, \dots, N$. We use uniform meshes which we label by "n" when they have 2^n elements.

We are not going to try to evaluate the negative-order norms of the errors e_u and e_q because negative-order norms are quite difficult to compute numerically. Instead, we are going to consider the integral of e_u on $(0,1)$. This is actually a sensible things to do, note that

$$\left| \int_0^1 \varphi \right| \leq \; C \inf_{s\geq 0} \| \varphi \|_{H^{-s}(0,1)},$$

which shows that the integrals converge with an order which cannot be smaller than that of *all* their negative-order norms.

In Table 1, we compare the case in which $\mathsf{D} = (0,1)$ with that in which $\mathsf{D} = (h, 1-h)$; in both cases we take $k_j = k$ for $j = 1, \ldots, N$. In the first case, we see that the pointwise error in u and q converges with order $k+1$, as predicted by Theorem 2.1 and that the numerical traces converge at least with order $2\,k+1$, as predicted by Theorem 2.2. The integral of the error in u also converges with order $2\,k+1$ as predicted by Theorem 2.3. When we change D to $(h, 1-h)$, we see that, as predicted by our theoretical results, the pointwise orders of convergence remain equal to $k+1$ whereas the orders for the remaining quantities become $k+2$.

In Table 2, we consider the case $\mathsf{D} = (h, 1-h)$ and investigate the effect of modifying the polynomial degrees k_1 and k_N. We first consider the case in which $k_1 = k_N = 2\,k-1$. We see that the order of convergence of the pointwise errors are still $k+1$, and that that of the numerical traces become at most $2\,k+1$. We then consider the case in which $k_1 = k_N = 2\,k-2$ and see that, although the orders of the pointwise errors remain $k+1$, those of the remaining quantities become $2\,k$ only. This shows that Theorem 2.3 and 2.2 are sharp and confirms the necessity of using polynomials of degree $2\,k-1$ in the extreme intervals in order to maintain the original order of convergence of the negative order norms and of the numerical traces.

6. Extensions and concluding remarks. Note that the choice of the HDG method to study this new approach is not essential; the approach can be readily applied to other numerical methods methods. In fact, other DG methods have been studied in [8].

The extension of the method to problems in several space dimension constitutes the subject of ongoing research. Preliminary results indicating that the method actually works very well have been obtained by one of the authors [8]. We would like to end this paper by displaying one of those results.

So, we consider the problem of finding an approximation to the solution of

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= u_D \text{ in } \partial\Omega, \end{aligned}$$

by using a straightforward extension of our method. We consider the simple case in which Ω is the unit circle and f and u_D are such that the exact solution in $u(x,y) = \sin(\rho(x^2+y^2))/\sin(\rho)$ where $\rho = 1.196704724$; note that $u_D \equiv 1$. We take D to be a polygonal subdomain of Ω and use an LDG method [6] with $C_{11} = 1$ to solve in D. A preliminary result is displayed in Fig. 6. We see that, even though the distance of D to Ω is a few times the diameter of a typical element, the approximation to u in the whole domain Ω seems very reasonable.

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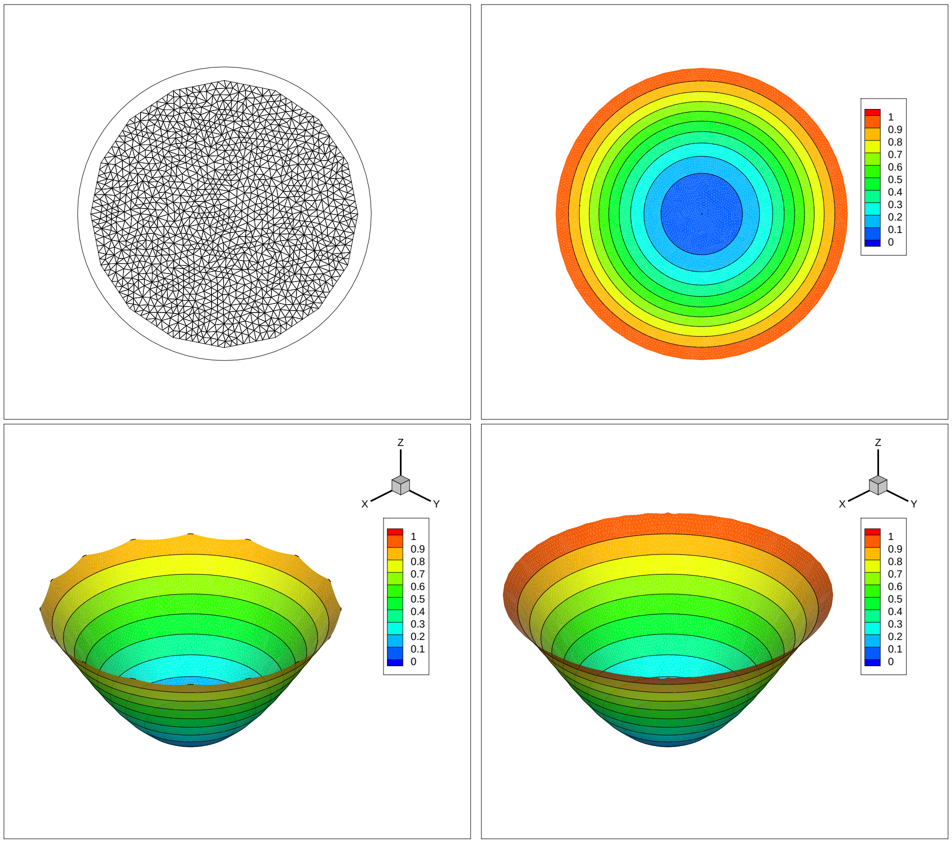

FIG. 1. *Domain and mesh (top, left) with the corresponding approximate solution (right column). Note that there is an annular subdomain of the exact domain which has not been meshed. The approximate solution therein is computed as an extension of the approximation in the meshed subdomain (bottom, left).*

Appendix. Here, we prove two results. The first concerns the smoothness of the Green's funtions and the second the approximability properties of the projections we are unsing in the analysis.

6.1. Smoothness of the Green's functions. Here we show that the Green's functions $G_x(y)$ and $\mathcal{G}_x(y)$ are smooth in the variable x too.

PROPOSITION 6.1. *If x is not one of the nodes x_j, $j = 0, \dots, N$, we have that*

$$\frac{\partial}{\partial x} G_x(y) = -\, c(x)\, \mathcal{G}_x(y),$$
$$\frac{\partial}{\partial x} \mathcal{G}_x(y) = -\, d(x)\, G_x(y),$$

for all $y \in (0,1)$, provided c and d are continuous at the nodes x_j, $j = 0, \dots, N$.

Proof. By formally taking derivatives with respect to x in the equations defining the Green's function G_x, (3.5), we get

$$\begin{aligned} -\left(\frac{1}{c}\,(\frac{\partial}{\partial x}G_x)'\right)' + d\,\frac{\partial}{\partial x}G_x &= 0 && \text{in } (0,x)\cup(x,1),\\ \frac{\partial}{\partial x}G_x &= 0 && \text{on } \{0,1\},\\ [\![\frac{\partial}{\partial x}G_x\, n]\!] &= -\,[\![(G_x)'\, n]\!] && \text{at } \{x\},\\ [\![\frac{1}{c}\,(\frac{\partial}{\partial x}G_x)'\, n]\!] &= -\,[\![(\frac{1}{c}\,(G_x)')'\, n]\!] && \text{at } \{x\}, \end{aligned}$$

and since

$$\begin{aligned} -\,[\![(G_x)'\, n]\!] &= \,-\,c(x)\,[\![\frac{1}{c}(G_x)'\, n]\!] = -c(x),\\ -\,[\![(\frac{1}{c}\,(G_x)')'\, n]\!] &= \,-\,d(x)\;\,[\![G_x]\!] = 0, \end{aligned}$$

by definition of G_x, we obtain the first identity. The second is proven in a similar way. This completes the proof. ☐

6.2. Proof of Proposition 4.1. To the knowledge of the authors, there is no proof of Proposition 4.1 in the available literature. For this reason, we include a detailed proof here.

Step 1: A scaling argument. By a simple scaling argument, it is enough to prove Proposition 4.1 for the interval $(\alpha,\beta)=(-1,1)$. To prove such result, we first establish a link between the projections $\pi_n^{\pm}$ and the L^2-projection, and then estimate the approximation error of the L^2-projection.

In what follows, we simplify our notation and write $\mathcal{I}_n^{\pm}$ and $\pi_n^{\pm}$ instead of $\mathcal{I}_{n,(-1,1)}^{\pm}$ and $\pi_{n,(-1,1)}^{\pm}$, respectively. We also write $\|\cdot\|$ instead of $\|\cdot\|_{L^\infty(-1,1)}$.

Step 2: The relation between the projections $\pi_n^{\pm}$ and the L^2-projection. We begin by relating the projection $\pi_n^{\pm} f$ with the L^2-projection $\mathsf{P}_n f$ into $\mathcal{P}_n(-1,1)$. This is a well known result, see [12], [10] and [11], but we include it here for the sake of completeness.

LEMMA 6.2. *We have that*

$$\pi_n^{\pm} f(x) = \mathsf{P}_n f(x) + (f-\mathsf{P}_n f)(\mp 1)\,(\mp 1)^n\,L_n(x),$$

for $x\in(-1,1)$.

Proof. If we assume that we can write

$$f(x) = \sum_{i=0}^{\infty} \omega_i\, L_i(x),$$

using the fact that $L_i(\pm 1) = (\pm 1)^i$, it is not difficult to see that

$$\pi^{\pm} f_n(x) = \sum_{i=0}^{n-1} \omega_i\, L_i(x) + \left(\sum_{i=n}^{\infty} \omega_i\,(\mp 1)^i\right)\,(\mp 1)^n\, L_n(x),$$

or, equivalently, that

$$\pi^{\pm} f(x) = \sum_{i=0}^{n} \omega_i \, L_i(x) + \left(\sum_{i=n+1}^{\infty} \omega_i \, (\mp 1)^i \right) (\mp 1)^n \, L_n(x).$$

Since

$$\mathsf{P}_n f(x) = \sum_{i=0}^{n} \omega_i \, L_i(x),$$

we see that

$$\pi^{\pm} f(x) = \mathsf{P}_n f(x) + (f - \mathsf{P}_n f)(\mp 1) \, (\mp 1)^n \, L_n(x).$$

This completes the proof. □

This allows us to obtain the following estimates.

LEMMA 6.3. *We have that*

$$\max\{ \, \| \mathfrak{I}_n^{\pm} f \| \, , \, \| \mathfrak{I}_n^{+} f - \mathfrak{I}_n^{-} f \| \, \} \leq \, 2 \, \| f - \mathsf{P}_n f \|.$$

Proof. By the definition of the approximation error $\mathfrak{I}_n^{\pm} f$ and the previous result, we have that

$$\mathfrak{I}_n^{\pm} f(x) = ((f - \mathsf{P}_n f)(x) - (f - \mathsf{P}_n f)(\mp 1) \, L_n(\mp 1) \, L_n(x),$$

and since $\| L_n \| \leq 1$, we obtain that

$$\| \mathfrak{I}_n^{\pm} f \| \leq \, 2 \, \| f - \mathsf{P}_n f \|.$$

This proves the bound for $\| \mathfrak{I}_n^{\pm} f \|$.

To prove the remaining bound, we are going to show that

$$\| \mathfrak{I}_n^{+} f - \mathfrak{I}_n^{-} f \| = | \, \mathfrak{I}_n^{+} f(1) \, | = | \, \mathfrak{I}_n^{-} f(-1) \, |;$$

the result will then immediately follow from the bound for $\| \mathfrak{I}_n^{\pm} f \|$. By definition of $\mathfrak{I}_n^{\pm}$, we have that, by Lemma 6.2,

$$\mathfrak{I}_n^{+} f(x) = \sum_{i=n+1}^{\infty} \omega_i \, L_i(x) - \left(\sum_{i=n+1}^{\infty} \omega_i \, (-1)^i \right) (-1)^n \, L_n(x),$$

$$\mathfrak{I}_n^{-} f(x) = \sum_{i=n+1}^{\infty} \omega_i \, L_i(x) - \left(\sum_{i=n+1}^{\infty} \omega_i \right) L_n(x),$$

and so

$$\mathfrak{I}_n^{+} f(x) - \mathfrak{I}_n^{-} f(x) = 2 \left(\sum_{j=0}^{\infty} \omega_{n+1+2\,j} \right) L_n(x).$$

This implies that

$$2 \left(\sum_{j=0}^{\infty} \omega_{n+1+2\,j} \right) = \mathfrak{I}_n^{+} f(1) = (-1)^{n+1} \, \mathfrak{I}_n^{-} f(-1).$$

The result now follows from the fact that $\| L_n \| = 1$. This completes the proof. $\square$

Step 3: Estimating the approximation error for the L^2-projection. The estimate we need is contained in the following result.

PROPOSITION 6.4. *If $f \in W^{k+1,\infty}(-1,1)$, then*

$$\| f - \mathsf{P}_n f \| \leq \mathsf{C}\,\mathsf{K}(2; n+1, k+1) \, \| f^{(k+1)} \|,$$

provided $n \geq k$, where C is independent of f, n and k.

To prove this proposition, we are going to use the following improved version of one of Jackson's estimates. In it, we use the traditional expression

$$E_n(f) := \inf_{p \in \mathcal{P}_n(-1,1)} \| f - p \|.$$

LEMMA 6.5. *If $f \in W^{k+1,\infty}(-1,1)$, then*

$$E_n(f) \leq \left(\frac{\pi}{2}\right)^{k+1} \frac{(n-k)!}{(n+1)!} \, \| f^{(k+1)} \|,$$

provided $n \geq k$.

Proof. By an improved version of Jackson's Theorem V, see page 147 of Cheney's book [2], we have that

$$E_n(f) \leq \left(\frac{\pi}{2}\right)^{k} \frac{(n-k+1)!}{(n+1)!} \, \| f^{(k)} \|,$$

provided $n \geq k$. By the definition of $E.(f^{(k)})$, this immediately implies that

$$E_n(f) \leq \left(\frac{\pi}{2}\right)^{k} \frac{(n-k+1)!}{(n+1)!} \, E_{n-k}(f^{(k)}).$$

The result follows from the fact that

$$\begin{aligned} E_{n-k}(f^{(k)}) &\leq \begin{cases} \| f^{(k+1)} \| & \text{if } n = k, \\ \left(\frac{\pi}{2}\right) \frac{1}{(n+1-k)} \| f^{(k+1)} \| & \text{if } n > k, \end{cases} \\ &\leq \left(\frac{\pi}{2}\right) \frac{1}{(n+1-k)} \| f^{(k+1)} \| \end{aligned}$$

for $n \geq k$. This completes the proof. $\square$

We are now ready to prove Proposition 6.4.

Proof. In 1913 Gronwall [7] proved that

$$\| f - \mathsf{P}_n f \| \leq \mathsf{C} \sqrt{n+1} \, \| f \|.$$

This implies that

$$\| f - \mathsf{P}_n f \| \leq \mathsf{C} \sqrt{n+1} \, E_n(f),$$

and by Lemma 6.5,

$$\begin{aligned} \| f - \mathsf{P}_n f \| &\leq \mathsf{C} \left(\frac{\pi}{2}\right)^{k+1} \frac{(n-k)!}{(n+1)!} \sqrt{(n+1)} \, \| f^{(k+1)} \|, \\ &= \mathsf{C}\,\mathsf{K}(2; n+1, k+1) \, \| f^{(k+1)} \|, \end{aligned}$$

by the definition of $\mathsf{K}(\cdot;\cdot,\cdot)$, (2.11). This completes the proof. $\square$

REFERENCES

[1] F. Celiker and B. Cockburn, *Superconvergence of the numerical traces of discontinuous Galerkin and hybridized mixed methods for convection-diffusion problems in one space dimension*, Math. Comp. **76** (2007), 67–96.

[2] E.W. Cheney, *Introduction to approximation theory*, McGraw-Hill, Inc., 1966.

[3] B. Cockburn, B. Dong, and J. Guzmán, *A superconvergent LDG-hybridizable Galerkin method for second-order elliptic problems*, Math. Comp., to appear.

[4] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, *Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems*, Submitted.

[5] B. Cockburn, J. Guzmán, and H. Wang, *Superconvergent discontinuous Galerkin methods for second-order elliptic problems*, Math. Comp., to appear.

[6] B. Cockburn and C.-W. Shu, *The local discontinuous Galerkin method for time-dependent convection-diffusion systems*, SIAM J. Numer. Anal. **35** (1998), 2440–2463.

[7] T.H. Gronwall, *On the degree of convergence of Laplace's series*, Trans. Amer. Math. Soc. (Mathematischen Annalen, vol. 74 (1913), pp.213-270) **15** (1914), 1–30.

[8] D. Gupta, *Boundary-conforming discontinuous Galerkin methods via extensions from subdomains*, Ph.D. thesis, School of Mathematics, University of Minnesota, 2007.

[9] T.H. Rivlin, *An introduction to the approximation of functions*, Dover Publications, Inc., New York, 1969.

[10] D. Schötzau, *hp-DGFEM for parabolic evolution problems - applications to diffusion and viscous incompressible fluid flow*, Ph.D. thesis, Swiss Federal Institute of Technology, Zürich, 1999, Dissertation No. 13041.

[11] D. Schötzau and C. Schwab, *Time discretization of parabolic problems by the hp-version of the Discontinuous Galerkin Finite Element Method*, SIAM J. Numer. Anal. **38** (2000), 837–875.

[12] V. Thomée, *Galerkin Finite Element Methods for parabolic equations*, Springer Verlag, 1997.

Table 1

History of convergence when $k_j = k$ for $j = 1, \ldots, N$.

		$\lVert e_u \rVert_{L^\infty(0,1)}$		$\left\lvert \int_0^1 e_u \right\rvert$		$\lVert \widehat{e}_u \rVert_\infty$		$\lVert e_q \rVert_{L^\infty(0,1)}$		$\lVert \widehat{e}_q \rVert_\infty$	
k	mesh	error	order	error	order	error	order	error	order	error	order
					The case $a = 0$ and $b = 1$.						
	3	1.18E-03	2.04	1.26E-05	3.01	4.13E-06	3.15	1.38E-03	2.08	2.80E-05	3.05
	4	2.89E-04	2.03	1.57E-06	3.01	2.46E-07	3.07	3.36E-04	2.04	3.44E-06	3.02
1	5	7.16E-05	2.01	1.95E-07	3.00	1.49E-08	3.04	8.26E-05	2.02	4.27E-07	3.01
	6	1.78E-05	2.01	2.44E-08	3.00	9.17E-09	3.02	2.05E-05	2.01	5.32E-08	3.01
	7	4.44E-06	2.00	3.05E-09	3.00	5.69E-10	3.01	5.11E-06	2.01	6.64E-09	3.00
	3	1.47E-05	3.02	1.69E-09	5.04	4.63E-10	5.00	2.30E-05	3.03	4.72E-09	5.01
	4	1.82E-06	3.01	5.21E-11	5.02	1.45E-11	4.99	2.85E-06	3.02	1.47E-10	5.01
2	5	2.27E-07	3.01	1.62E-12	5.01	4.52E-13	5.00	3.54E-07	3.01	4.58E-12	5.00
	6	2.83E-08	3.00	5.04E-14	5.00	1.41E-14	5.00	4.41E-08	3.00	1.43E-13	5.00
	7	3.53E-09	3.00	1.57E-15	5.00	4.41E-16	5.00	5.50E-09	3.00	4.47E-15	5.00
	3	1.32E-07	4.02	1.56E-13	7.00	3.03E-14	7.20	1.55E-07	4.09	3.43E-13	7.02
	4	8.22E-09	4.01	1.22E-15	7.00	1.10E-16	7.11	9.40E-09	4.05	2.66E-15	7.01
3	5	5.12E-10	4.00	9.49E-18	7.00	4.09E-18	7.07	5.78E-10	4.02	2.07E-17	7.01
	6	3.20E-11	4.00	7.41E-20	7.00	1.56E-20	7.03	3.58E-11	4.01	1.61E-19	7.00
	7	2.00E-12	4.00	5.79E-22	7.00	6.03E-22	7.02	2.23E-12	4.01	1.26E-21	7.00
	3	9.24E-10	5.01	6.43E-18	9.03	1.77E-18	8.99	1.42E-09	5.02	1.81E-17	9.00
	4	2.88E-11	5.01	1.25E-20	9.01	3.48E-21	8.99	4.40E-11	5.01	3.52E-20	9.00
4	5	8.97E-13	5.00	2.43E-23	9.01	6.78E-24	9.00	1.37E-12	5.01	6.87E-23	9.00
	6	2.80E-14	5.00	4.73E-26	9.00	1.33E-26	9.00	4.27E-14	5.00	1.34E-25	9.00
	7	8.74E-16	5.00	9.23E-29	9.00	2.59E-29	9.00	1.33E-15	5.00	2.62E-28	9.00
					The case $a = h$ and $b = 1 - h$.						
	3	1.14E-03	2.36	8.54E-05	1.09	4.40E-04	2.77	7.12E-04	1.96	1.84E-05	4.03
	4	2.90E-04	2.33	2.14E-05	2.35	8.03E-05	2.89	2.21E-04	1.99	1.35E-06	4.45
1	5	7.03E-05	2.23	3.79E-06	2.72	1.23E-05	2.95	6.23E-05	1.99	1.63E-07	3.32
	6	1.75E-05	2.09	5.66E-07	2.87	1.71E-06	2.98	1.66E-05	2.00	3.47E-08	2.34
	7	4.40E-06	2.04	7.73E-08	2.94	2.26E-07	2.99	4.27E-06	2.00	5.40E-09	2.74
	3	2.23E-05	3.52	4.15E-06	3.29	1.50E-05	3.75	4.10E-04	2.85	2.36E-05	3.73
	4	2.83E-06	3.52	4.70E-07	3.70	1.56E-06	3.85	7.37E-05	2.92	2.44E-06	3.86
2	5	3.17E-07	3.44	4.08E-08	3.84	1.28E-07	3.92	1.12E-05	2.96	2.00E-07	3.93
	6	3.50E-08	3.32	3.03E-09	3.92	9.29E-09	3.96	1.56E-06	2.98	1.44E-08	3.96
	7	3.99E-09	3.20	2.07E-10	3.96	6.26E-10	3.98	2.05E-07	2.99	9.71E-10	3.98
	3	5.06E-07	4.69	1.65E-07	5.00	4.61E-07	4.73	1.72E-05	3.93	8.82E-07	5.03
	4	3.05E-08	4.50	8.70E-09	5.00	2.63E-08	4.87	1.67E-06	3.97	4.61E-08	5.02
3	5	1.46E-09	4.39	3.62E-10	5.00	1.13E-09	4.94	1.32E-07	3.99	1.90E-09	5.01
	6	6.47E-11	4.15	1.31E-11	5.00	4.20E-11	4.97	9.34E-09	3.99	6.87E-11	5.01
	7	2.93E-12	4.04	4.43E-13	5.00	1.43E-12	4.99	6.22E-10	4.00	2.31E-12	5.00
	3	1.06E-08	4.80	4.79E-09	6.02	1.03E-08	5.74	4.54E-07	4.86	1.56E-08	5.69
	4	3.47E-10	4.82	1.39E-10	6.02	3.31E-10	5.85	2.51E-08	4.93	5.04E-10	5.84
4	5	8.34E-12	4.86	3.00E-12	6.04	7.68E-12	5.92	1.07E-09	4.96	1.17E-11	5.92
	6	1.72E-13	4.86	5.50E-14	6.03	1.48E-13	5.96	3.92E-11	4.98	2.24E-13	5.96
	7	3.37E-15	4.90	9.31E-16	6.02	2.57E-15	5.98	1.33E-12	4.99	3.89E-15	5.98

Table 2

History of convergence when $a = h$ and $b = 1 - h$, and $k_j = k$ for $j = 2, \dots, N - 1$.

		$\| e_u \|_{L^\infty(0,1)}$		$\left\| \int_0^1 e_u \right\|$		$\| \widehat{e}_u \|_\infty$		$\| e_q \|_{L^\infty(0,1)}$		$\| \widehat{e}_q \|_\infty$	
k	mesh	error	order	error	order	error	order	error	order	error	order
					The case $k_1 = k_N = 2\,k - 1$.						
	3	1.14E-03	2.36	8.54E-05	1.09	4.40E-04	2.77	7.12E-04	1.96	1.84E-05	4.03
	4	2.90E-04	2.33	2.14E-05	2.35	8.03E-05	2.89	2.21E-04	1.99	1.35E-06	4.45
1	5	7.03E-05	2.23	3.79E-06	2.72	1.23E-05	2.95	6.23E-05	1.99	1.63E-07	3.32
	6	1.75E-05	2.09	5.66E-07	2.87	1.71E-06	2.98	1.66E-05	2.00	3.47E-08	2.34
	7	4.40E-06	2.04	7.73E-08	2.94	2.26E-07	2.99	4.27E-06	2.00	5.40E-09	2.74
	3	7.42E-06	2.90	1.64E-07	5.01	1.40E-05	4.83	1.72E-05	3.93	8.82E-07	5.03
	4	1.28E-06	2.99	8.67E-09	5.00	1.40E-06	4.92	1.67E-06	3.97	4.61E-08	5.02
2	5	1.89E-07	3.00	3.61E-10	5.00	1.13E-07	4.96	1.87E-07	3.44	1.90E-09	5.01
	6	2.58E-08	3.00	1.31E-11	5.00	8.03E-09	4.98	2.57E-08	2.99	6.88E-11	5.01
	7	3.37E-09	3.00	4.42E-13	5.00	5.37E-10	4.99	3.36E-09	3.00	2.31E-12	5.00
	3	5.41E-08	4.00	2.41E-11	5.00	2.32E-10	6.71	5.36E-08	3.99	4.51E-10	7.03
	4	5.13E-09	4.01	7.89E-13	5.81	4.09E-12	6.87	5.12E-09	4.00	7.29E-12	7.02
3	5	4.02E-10	4.00	1.20E-14	6.57	4.96E-14	6.94	4.01E-10	4.00	8.43E-14	7.01
	6	2.83E-11	4.00	1.31E-16	6.81	4.88E-16	6.97	2.82E-11	4.00	8.08E-16	7.01
	7	1.88E-12	4.00	1.21E-18	6.91	4.28E-18	6.99	1.88E-12	4.00	7.01E-18	7.00
	3	3.00E-10	4.94	2.46E-14	9.07	6.81E-14	8.71	3.02E-10	5.00	1.34E-13	9.02
	4	1.59E-11	4.99	1.21E-16	9.04	3.72E-16	8.86	1.59E-11	5.00	6.67E-16	9.02
4	5	6.62E-13	5.00	3.90E-19	9.02	1.27E-18	8.94	6.62E-13	5.00	2.16E-18	9.01
	6	2.40E-14	5.00	9.91E-22	9.01	3.30E-21	8.97	2.40E-14	5.00	5.51E-21	9.01
	7	8.09E-16	5.00	2.22E-24	9.00	7.48E-24	8.98	8.09E-16	5.00	1.23E-23	9.00
					The case $k_1 = k_N = 2\,k - 2$.						
	3	1.14E-03	2.36	8.54E-05	1.09	4.40E-04	2.77	7.12E-04	1.96	1.84E-05	4.03
	4	2.90E-04	2.33	2.14E-05	2.35	8.03E-05	2.89	2.21E-04	1.99	1.35E-06	4.45
1	5	7.03E-05	2.23	3.79E-06	2.72	1.23E-05	2.95	6.23E-05	1.99	1.63E-07	3.32
	6	1.75E-05	2.09	5.66E-07	2.87	1.71E-06	2.98	1.66E-05	2.00	3.47E-08	2.34
	7	4.40E-06	2.04	7.73E-08	2.94	2.26E-07	2.99	4.27E-06	2.00	5.40E-09	2.74
	3	2.23E-05	3.52	4.15E-06	3.29	1.50E-05	3.75	4.10E-04	2.85	2.36E-05	3.73
	4	2.83E-06	3.52	4.70E-07	3.70	1.56E-06	3.85	7.37E-05	2.92	2.44E-06	3.86
2	5	3.17E-07	3.44	4.08E-08	3.84	1.28E-07	3.92	1.12E-05	2.96	2.00E-07	3.93
	6	3.50E-08	3.32	3.03E-09	3.92	9.29E-09	3.96	1.56E-06	2.98	1.44E-08	3.96
	7	3.99E-09	3.20	2.07E-10	3.96	6.26E-10	3.98	2.05E-07	2.99	9.71E-10	3.98
	3	6.12E-08	4.21	4.79E-09	6.02	1.03E-08	5.74	4.54E-07	4.86	1.56E-08	5.69
	4	5.33E-09	4.15	1.39E-10	6.02	3.31E-10	5.85	2.51E-08	4.93	5.04E-10	5.84
3	5	4.06E-10	4.05	3.00E-12	6.04	7.68E-12	5.92	1.07E-09	4.96	1.17E-11	5.92
	6	2.83E-11	4.01	5.50E-14	6.03	1.48E-13	5.96	3.92E-11	4.98	2.24E-13	5.96
	7	1.88E-12	4.00	9.31E-16	6.02	2.57E-15	5.98	1.88E-12	4.48	3.89E-15	5.98
	3	2.96E-10	4.88	1.13E-12	7.10	3.86E-12	7.73	3.07E-10	6.33	5.76E-12	7.67
	4	1.59E-11	4.98	1.24E-14	7.68	3.84E-14	7.85	1.60E-11	5.03	5.77E-14	7.83
4	5	6.62E-13	5.00	8.50E-17	7.83	2.50E-16	7.92	6.62E-13	5.01	3.75E-16	7.92
	6	2.40E-14	5.00	4.46E-19	7.92	1.27E-18	7.96	2.40E-14	5.00	1.91E-18	7.96
	7	8.09E-16	5.00	2.03E-21	7.96	5.71E-21	7.98	8.09E-16	5.00	8.57E-21	7.98