CYCLIC SIEVING AND PROMOTION

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The Cyclic Sieving Phenomenon

X = finite set.

$C = \langle c \rangle$ = finite cyclic group acting on X, $\zeta \in \mathbb{C}$ = primitive $|C|^{th}$ root of unity.

$X(q) \in \mathbb{Z}[q]$ = polynomial in q.

Definition: $(X, C, X(q))$ exhibits the *cyclic sieving phenomenon* (CSP) if for all $d \geq 0$ we have $|X^{c^d}| = X(\zeta^d)$. ($\Rightarrow |X| = X(1)$)

Example of CSP

$$X := \ \{ \ k\text{-subsets of } [n] \ \}$$

$$C := \mathbb{Z}/n\mathbb{Z} \text{ acting on } X \text{ via } (1, 2, \ldots, n)$$

For $n = 4$,

$$\{1, 2\} \rightsquigarrow \{2, 3\} \rightsquigarrow \{3, 4\} \rightsquigarrow \{1, 4\}$$

$$\{1, 3 \rightsquigarrow \{2, 4\}$$

Theorem: [Reiner, Stanton, White 2004)] $(X, C, X(q))$ exhibits the CSP, where

$$X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = q\text{-binomial coefficient}$$

Notation: q-analogues

Recall...

$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}.$

$[n]!_q := [n]_q[n-1]_q \cdots [1]_q.$

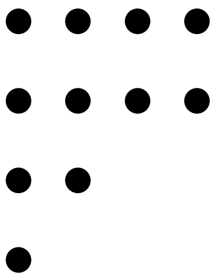
$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n!]_q}{[k!]_q[(n-k)!]_q}.$

CSP paradigm: $X(q)$ is the 'q-analogue' of a formula counting $|X|$...

Review of Partitions

A *partition* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ *of* n is a weakly decreasing sequence of positive integers such that $\lambda_1 + \cdots + \lambda_k = n$. (write $\lambda \vdash n$)

Partitions have *diagrams*. For example, the diagram of $(4, 4, 2, 1)$ is



$|\lambda| :=$ number of nodes in the diagram of λ.

A partition is *rectangular* if its diagram is.

Review of Tableaux

If $\lambda \vdash n$, a λ-*tableau* is an assignment of a positive integer to every node of the diagram of λ.

A possible (4,4,2,1)-tableau: $\begin{array}{llll} 1 & 1 & 3 & 4 \\ 3 & 4 & 5 & 5 \\ 6 & 7 & & \\ 7 & & & \end{array}$

A tableau is *semistandard* if its entries increase strictly down columns and weakly across rows.

Let $CST(\lambda, k)$ be the set of semistandard λ-tableaux with entries $\leq k$. (The above tableau is in $CST((4, 4, 2, 1), 7)$.

A tableau is *standard* if it is semistandard and a bijection onto $\{1, 2, \ldots, |\lambda|\}$. $SYT(\lambda)$ is the set of standard tableaux of shape λ.

Jeu-de-Taquin Promotion

Given a tableau $T \in CST(\lambda, k)$, we define another tableau $j(T) \in CST(\lambda, k)$...

1. Replace every $k \in T$ with a dot.

2. Play jeu-de-taquin to move the dots to the upper left hand corner.

3. Increase every entry of T by 1.

4. Replace every dot with a 1. The resulting tableau is $j(T)$.

NOTE: The definition of $j(T)$ for $T \in CST(\lambda, k)$ depends on k.

Example of Promotion

Let T be the following element of $CST((4,4,2,1),7)$.

$$T = \begin{array}{llll} 1 & 1 & 3 & 4 \\ 3 & 4 & 5 & 5 \\ 6 & 7 & & \\ 7 & & & \end{array}$$

To find $j(T)$:

$$\begin{array}{llll} 1 & 1 & 3 & 4 \\ 3 & 4 & 5 & 5 \\ 6 & \bullet & & \\ \bullet & & & \end{array} \rightsquigarrow \begin{array}{llll} \bullet & 1 & 3 & 4 \\ 1 & 4 & 5 & 5 \\ 3 & \bullet & & \\ 6 & & & \end{array} \rightsquigarrow \begin{array}{llll} \bullet & \bullet & 3 & 4 \\ 1 & 1 & 5 & 5 \\ 3 & 4 & & \\ 6 & & & \end{array} \rightsquigarrow \begin{array}{llll} \bullet & \bullet & 4 & 5 \\ 2 & 2 & 6 & 6 \\ 4 & 5 & & \\ 7 & & & \end{array} \rightsquigarrow \begin{array}{llll} 1 & 1 & 4 & 5 \\ 2 & 2 & 6 & 6 \\ 4 & 5 & & \\ 7 & & & \end{array} = j(T)$$

CSP for Standard Tableaux

Observation: j restricts to a map $SYT(\lambda) \to SYT(\lambda)$.

Theorem: [R. 2007] Let $\lambda \vdash n$ be a *rectangular* partition. The triple $(X, C, X(q))$ exhibits the CSP, where:

$$X = SYT(\lambda)$$

$$C = \mathbb{Z}/n\mathbb{Z} \text{ acting by promotion}$$

$$X(q) = \frac{[n]!_q}{\Pi_{(i,j)\in\lambda}[h_{ij}]_q} = q\text{-hook length formula}$$

Some Implications and Remarks

Corollary: For $\lambda \vdash n$ rectangular, j^n fixes every element of $SYT(\lambda)$.

Corollary: For $\lambda \vdash n$ rectangular, $|SYT(\lambda)|$ is counted by the hook length formula:

$$|SYT(\lambda)| = \frac{n!}{\Pi_{(i,j)\in\lambda} h_{ij}}.$$

Remark: This is in general false for λ not rectangular. $\lambda = (3,2) \vdash 5$ is a counterexample.

An example: $\lambda = (3,3)$

The action of j on $SYT((3,3))$:

$$\begin{array}{ccc}1&2&3\\4&5&6\end{array} \rightsquigarrow \begin{array}{ccc}1&3&4\\2&5&6\end{array} \rightsquigarrow \begin{array}{ccc}1&2&5\\3&4&6\end{array}$$

$$\begin{array}{ccc}1&3&5\\2&4&6\end{array} \rightsquigarrow \begin{array}{ccc}1&2&4\\3&5&6\end{array}$$

$$X(q) = \frac{[6]!_q}{[4]_q[3]_q^2[2]_q^2[1]_q}$$

We compute, for $\zeta = e^{\pi i/3}$:

$$\begin{array}{lll} X(\zeta^0) = 5 & X(\zeta^1) = 0 & X(\zeta^2) = 2 \\ X(\zeta^3) = 3 & X(\zeta^4) = 2 & X(\zeta^5) = 0 \end{array}$$

Application: Handshake Patterns

Suppose $2n$ people are sitting around a circular table. A *handshake pattern* is a way for all of the people to shake hands in such a way that no one crosses arms.

Example: The handshake patterns for $n = 3$.

Let H_n be the set of handshake patterns with $2n$ people.

$|H_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$.

Handshake Patterns and $2 \times n$ Standard Tableaux

There is a bijection due to White between H_n and $SYT(n^2)$.

Example:

Cyclic Sieving and Handshake Patterns

$C = \mathbb{Z}/2n\mathbb{Z}$ acts on H_n by cyclic rotation of the table.

Fact: [White] Under White's bijection, rotation ↔ promotion.

Corollary: (Proved directly by White, Heitsch) The triple $(H_n, C, X(q))$ exhibits the CSP, where

$$X(q) = \frac{[2n]_q[2n-1]_q \cdots [n+2]_q}{[n]!_q}.$$

Review of Schur functions

Fix $k \geq 0$. Given T a tableau with entries $\leq k$, $\mathrm{cont}(T)$ (the *content*) of T is (1's in T, 2's in T, $\ldots$, k's in T) $\in \mathbb{N}^k$.

$$T = \begin{array}{llll} 1 & 1 & 3 & 4 \\ 3 & 4 & 5 & 5 \\ 6 & 7 & & \\ 7 & & & \end{array} \Rightarrow \mathrm{cont}(T) = (2,0,2,2,2,1,2)$$

The *Schur function* (in k variables) associated to λ is

$$s_\lambda(x_1,\ldots,x_k) = \sum_{T \in CST(\lambda,k)} x^{\mathrm{cont}(T)} \in \mathbb{Z}[x_1,\ldots,x_k].$$

CSP for Semistandard Tableaux

Theorem: [R. 2007] Let $\lambda \vdash n$ be a *rectangular* partition and let $k \geq 0$. The triple $(X, C, X(q))$ exhibits the CSP, where:

$$X = CST(\lambda, k)$$

$$C = \mathbb{Z}/k\mathbb{Z} \text{ acting by promotion}$$

$$X(q) = s_\lambda(1, q, q^2, \ldots, q^{k-1}).$$

Remark: This is in general false for λ not rectangular.

Proofs: Representation Theory

Q: How do we prove these CSPs?

A: We define *representations* with bases indexed by X ($= SYT(\lambda)$ or $CST(\lambda, k)$ for λ rectangular) such that our cyclic action (promotion) is (essentially) in the image of the representation.

Q: What 'natural' module has degree $|SYT(\lambda)| = f^{\lambda}$?

A: The irreducible S_n-module Specht$^{\lambda}$. We will use the Kazhdan-Lusztig cellular avatar of this representation.

Kazhdan-Lusztig Polynomials

For any $u, v \in S_n$, we have the Kazhdan-Lusztig polynomial $P_{u,v}(q) \in= \mathbb{N}[q]$.

Fact: $P_{u,v}(q) \equiv 0$ iff $u \not\leq v$ in Bruhat.

Fact: $\deg(P_{u,v}(q)) \leq \frac{\ell(u)-\ell(v)-1}{2}$.

$\mu(u, v) = [q^{\frac{\ell(u)-\ell(v)-1}{2}}]P_{u,v}(q)$.

$\mu[u, v] = \max(\mu(u, v), \mu(v, u))$.

Tableaux Descent Sets

Given $T \in SYT(\lambda)$ for $\lambda \vdash n$ and $1 \leq i < n$, i is a *descent* of T if $i+1$ occurs strictly south and weakly west of i in T.

$$D(T) = \{i \in [n-1] \mid i \text{ is a descent of } T\}.$$

Example:

The KL cellular representation

Let $\lambda \vdash n$. For $P, Q \in SYT(\lambda)$, set $\mu[P,Q] := \mu[read(P), read(Q)]$, where $read(T)$ is the column reading word of T.

Let $V^\lambda := \mathbb{C}[SYT(\lambda)]$ and define an S_n-action on V^λ by:

$$s_i P = \begin{cases} -P & \text{if } i \in D(P) \\ P + \sum_{i \in D(Q)} \mu[P,Q]Q & \text{if } i \notin D(P) \end{cases}$$

Fact: V^λ is isomorphic to the Specht module of shape λ. The representation V^λ is the *(left) KL cellular representation* of shape λ.

The Long Cycle in the Cellular Representation

Let $c = (1, 2, \ldots, n) \in S_n$ be the long cycle.

Theorem: [R. 2007] Let $\lambda = b^a \vdash n$ be a *rectangular* partiton and let $\rho : S_n \to V^\lambda$ be the KL cellular representation of shape λ with basis $\{e_U \,|\, U \in SYT(\lambda)\}$. We have that

$$\rho(c)(e_U) = (-1)^{a-1} e_{j(U)}.$$

Proof uses the fact that λ is rectangular:

For any $\mu \vdash n$, the restriction $\text{Specht}^\mu \downarrow_{S_{n-1}}^{S_n}$ is irreducible if and only if μ is a rectangle.

Semistandard Tableaux

Q: For $k \geq 0$, what 'natural' module has degree $|CST(\lambda, k)|$?

A: The irreducible $GL_k(\mathbb{C})$ module $\mathbb{S}^\lambda(\mathbb{C}^k)$, where $\mathbb{S}^\lambda$ is the Schur functor corresponding to the partition λ.

Q: What order k element of $GL_k(\mathbb{C})$ can model promotion on $CST(\lambda, k)$?

A: The long cycle $(1, 2, \ldots, k) \in S_k \subset GL_k(\mathbb{C})$.

Q: What basis should we take for $\mathbb{S}^\lambda(\mathbb{C}^k)$?

The Canonical Basis

Work of Dirnfeld, Du, Jimbo, Kashiwara and Lusztig led to a canonical basis for $U_q(\mathfrak{sl}_k(\mathbb{C}))$ which specializes at $q = 1$ to a basis B for the universal enveloping algebra $U(\mathfrak{sl}_k(\mathbb{C}))$.

Fact: Let $V(\lambda)$ be the $\mathfrak{sl}_k$ irrep with highest weight λ. Then, $B \cap V(\lambda)$ is a basis for $V(\lambda)$ called the *canonical basis*.

Bernstein and Zelevinsky use a 'string paramaterization' of the canonical basis in work relating to evacuation based $q = -1$ phenomena.

However, for our purposes, this framework is too complicated to effectively work with...

The Dual Canonical Basis

Let $x = (x_{i,j})_{1 \leq i,j \leq k}$ be a $k \times k$ matrix of indeterminates.

We have a dual canonical basis of $\mathcal{O}_q(SL_k(\mathbb{C}))$ which specializes at $q = 1$ to a basis of $\mathcal{O}(SL_k(\mathbb{C}))$. There is a closely related basis of $\mathbb{C}[x_{11}, \ldots, x_{kk}]$ also called the dual canonical basis.

Q: What basis for $\mathbb{S}^\lambda(\mathbb{C}^k)$ should we take to get our CSP?

A: A homomorphic image of a subset of the dual canonical basis. (Uses Skandera's characterization of the DCB in terms of immanants.)

Semistandard Module Result

Let $\lambda \vdash n$ be rectangular and $k \geq 0$. Let U^{λ} be the realization of $\mathbb{S}^{\lambda}(\mathbb{C}^{k})$ alluded to above. Denote this representation by ρ.

Write our basis for U^{λ} as $\{f_T \,|\, T \in CST(\lambda, k)\}$.

Theorem: [R. 2007] Let $d = (1, 2, \ldots, k) \in S_k$. Then, for some predictable function $\gamma : CST(\lambda, k) \to \{1, -1\}$ depending on λ,

$$\rho(d)(f_T) = \gamma(T)(f_{j(T)})$$

for all $T \in CST(\lambda, k)$.

Dihedral Actions

We have Schützenberger evacuation $e : CST(\lambda, k) \to CST(\lambda, k)$ and also $e : SYT(\lambda) \to SYT(\lambda)$.

Fact: For rectangular $\lambda \vdash n$, the operators e and j satisfy

$$\begin{aligned} e^2 &= 1 \\ j^n &= 1 \qquad \text{on } SYT(\lambda) \\ j^k &= 1 \qquad \text{on } CST(\lambda, k) \\ eje &= j^{-1}. \end{aligned}$$

So, e and j generate a dihedral group D.

Can we say anything about the sizes of the fixed sets under the action of elements of D?

Berenstein-Zelevinsky Evacuation

Theorem: [Berenstein-Zelevinsky 1996] Let $\lambda \vdash n$ be an *arbitrary* partition and let w_o be the long element of S_n. Let ρ be the KL cellular representation with basis $\{g_P \,|\, P \in SYT(\lambda)\}$. Then,

$$\rho(w_o)(g_P) = c(g_{e(P)}),$$

where $c = 1$ or -1 and depends on λ.

A similar result holds for semistandard tableaux...

Use these results to get our enumeration of fixed points.

Dihedral Actions on Tableaux

Theorem: [R. 2007] Let $\lambda \vdash n$ be rectangular. The number of fixed points of e on $SYT(\lambda)$ is

$$|\chi^{\lambda}(w_o)|.$$

The number of fixed points of ej on $SYT(\lambda)$ is

$$|\chi^{\lambda}(w_o c)|.$$

A similar result holds for semistandard tableaux...

Handshake Patterns Revisited: Dihedral Sieving

D_{2n} acts on H_n by cyclic rotation and reflecting the table about the vertical axis:

Fact: Under White's bijection, reflection ↔ evacuation.

Corollary: The number of fixed points in H_n under the action of reflection is

$$|\chi^{n^2}(w_o)|.$$

The number of fixed points in H_n under the composition of the actions of reflection and rotation is

$$|\chi^{n^2}(cw_o)|.$$