KAZHDAN-LUSZTIG IMMANANTS AND THE CAYLEY-HAMILTON THEOREM

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Abstract. We give a generalization of the Cayley-Hamilton Theorem using Kazhdan-Lusztig immanants.

1. Introduction

In [2] the authors define the Kazhdan-Lusztig immanants using the Kazhdan-Lusztig basis of the symmetric group algebra $\mathbb{C}[S_n]$. The study of the cone generated by these immanants has lead to several combinatorial tests for determining when linear combinations of products of matrix minors are totally nonnegative or Schur nonnegative [2]. These tests have been used by Lam, Postnikov, and Pylyavskyy to resolve several conjectures in Schur positivity [1]. The determinant is a special case of a Kazhdan-Lusztig immanant, so it is natural to ask whether and how linear algebra results about the determinant generalize to these objects. In [2] and [3] the authors develop some results of this flavor. In this vein, we present a generalization of the Cayley-Hamilton theorem.

2. Main

Fix a positive integer n. Let $x = (x_{ij})_{1 \leq i,j \leq n}$ be a general $n \times n$ matrix. For $w \in S_n$, define the *w-Kazhdan-Lusztig immanant* to be the element $\mathrm{Imm}_w(x)$ of the polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{nn}]$ given by

$$\mathrm{Imm}_w(x) = \sum_{v \in S_n} (-1)^{\ell(w) - \ell(v)} P_{w_o * v, w_o * w}(1) x_{1,v(1)} \cdots x_{n,v(n)}.$$

Here w_o is the long element of S_n and $P_{u,y}(q) \in \mathbb{N}[q]$ is the Kazhdan-Lusztig polynomial corresponding to the permutations u and v. It is possible to show that $\mathrm{Imm}_1(x) = \det(x)$, so that the Kazhdan-Lusztig immanants generalize the determinant.

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Given $m, n \in \mathbb{N}$ and permutations $u \in S_m$ and $v \in S_n$, we may define a permutation $u \oplus v \in S_{m+n}$ by letting u act on the first m letters of $[m+n]$ and letting v act on the last n letters. We quote the following property of Kazhdan-Lusztig immanants.

Lemma 2.1. [2, Proposition 6.5] *Suppose that $m, n \in \mathbb{N}$ and u and v are permutations in S_m and S_n, respectively. Suppose A is a block diagonal complex matrix with $diag(A) = (B, C)$ and $B \in M_m(\mathbb{C})$, $C \in M_n(\mathbb{C})$. We have that*

$$\mathrm{Imm}_{u \oplus v}(A) = \mathrm{Imm}_u(B)\mathrm{Imm}_v(C).$$

This lemma is the main tool for proving our Cayley-Hamilton type result. For the rest of the paper, fix a complex vector space V of dimension n and a linear operator $T : V \to V$. Given an ordered basis β of V and $w \in S_n$, define the (w, β)-polynomial $f_{w,\beta}(X) \in \mathbb{C}[X]$ of T by

$$f_{w,\beta}(X) = \mathrm{Imm}_w(I_n X - [T]_\beta).$$

Here I_n is the $n \times n$ identity matrix. $f_{w,\beta}(X)$ is the analogue of the characteristic polynomial in what follows.

It is easy to check that for permutations w other than the identity the Kazhdan-Lusztig immanant $\mathrm{Imm}_w(x)$ is not basis invariant, so the polynomial $f_{w,\beta}(X)$ does in general depend on the ordered basis β. As such, we are somewhat careful about the basis we use in stating our result.

Suppose that the invariant factors $p_1, \ldots, p_r$ of T have degrees $d_1 \leq \cdots \leq d_r$. We have a rational canonical basis $\gamma = (\gamma_1, \ldots, \gamma_r)$ of V, where $gamma_i$ corresponds to the factor p_i for all i. Fix an integer $s \in [r]$. Call an ordered basis β of V $s-$well behaved if β decomposes as $\beta = (\alpha, \gamma'_s, \alpha')$, where $\mathrm{span}(\alpha) = \mathrm{span}(\gamma_1 \cup \cdots \cup \gamma_{s-1})$, $\mathrm{span}(\gamma'_s) = \mathrm{span}(\gamma_s)$, and $\mathrm{span}(\alpha') = \mathrm{span}(\gamma_{s+1} \cup \cdots \cup \gamma_r)$.

Theorem 2.2. *Let $\beta = (\alpha, \gamma_s, \alpha')$ be an $s-$well behaved ordered basis of V. Let $u \in S_{|\alpha|}$ and $v \in |S_{\alpha'}|$. We have that*

$$rank(f_{u \oplus 1 \oplus v, \beta}(T)) \leq |\alpha'| = d_{s+1} + \ldots d_r.$$

Proof. Make V into a $\mathbb{C}[X]-$module by letting X act by the operator T. This gives V the structure of a finitely generated module over a Principal Ideal Domain, so we may apply the structure theorem for such modules to write

$$V \cong \bigoplus_{i=1}^{r} \mathbb{C}[X]/(p_i),$$

where the isomorphism is an isomorphism of $\mathbb{C}[X]$-modules. Moreover, we have the chain $p_1 | p_2 | \ldots | p_r$.

Using the Lemma and the facts that β is $s-$well behaved and $\mathrm{Imm}_1(x) = \det(x)$ we get that

$$\begin{aligned} f_{u\oplus 1\oplus v,\beta}(X) &= \mathrm{Imm}_{u\oplus 1\oplus v}(I_nX - [T]_\beta) \\ &= \mathrm{Imm}_{u\oplus 1\oplus v}(diag(A, B, C)) \\ &= \mathrm{Imm}_u(A)\det(B)\mathrm{Imm}_v(C) \\ &= g(X) * p_s(X) * h(X). \end{aligned}$$

That is, $f_{u\oplus 1\oplus v,\beta}(X)$ contains $p_s(X)$ as a factor. This means that $f_{u\oplus 1\oplus v,\beta}(X)$ kills every vector in the subspaces $\mathbb{C}[X]/(p_i)$ for $i \leq s$. The result follows from a dimension count and the definition of the action of X on V.

□

Since $\mathrm{Imm}_1(x) = \det(x)$ and the determinant is basis independent, this theorem reduces to the Cayley-Hamilton theorem with base field $\mathbb{C}$ if one takes u and v to be identity permutations. It would be interesting to see if versions of this theorem hold with weaker conditions on the ordered basis β.

Acknowledgements

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References

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[2] B. Rhoades and M. Skandera. Kazhdan-Lusztig immanants and products of matrix minors. *J. Algebra*, **304**, 2 (2006) pp. 793–811.

[3] B. Rhoades and M. Skandera. Kazhdan-Lusztig immanants and products of matrix minors, II (2006).