KAZHDAN-LUSZTIG IMMANANTS AND PRODUCTS OF MATRIX MINORS, II

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Abstract. We show that for each permutation w containing no decreasing subsequence of length k, the Kazhdan-Lusztig immanant $\mathrm{Imm}_w(x)$ vanishes on all matrices having k equal rows or columns. Also, we define two filtrations of the vector space of immanants via products of matrix minors and pattern avoidance and use the above result to show that these filtrations are equivalent. Finally, we construct new and simple inequalities satisfied by the minors of totally nonnegative matrices.

1. Introduction and Preliminaries

The Kazhdan-Lusztig basis $\{C'_w(q) \mid w \in S_n\}$ of the Hecke algebra $H_n(q)$, originally introduced in [14], has seen several applications in combinatorics and positivity. In [21] the authors define the Kazhdan-Lusztig immanants via the Kazhdan-Lusztig basis and obtain various positivity results concerning linear combinations of products of matrix minors. These results illuminate inequalities [10, Thm. 4.6] satisfied by the minors of certain matrices. (See, e.g., [2], [20], [23].) In addition, [21, Thm. 9] implies inequalities [15, Thm. 10] satisfied by certain symmetric functions. The inequalities in turn are used in [15] to revolve several conjectures in Schur positivity. In this paper, we further develop algebraic properties of the Kazhdan-Lusztig immanants and apply these immanants to obtain additional positivity results.

Fix $n \in \mathbb{N}$ and let $x = (x_{ij})_{1 \leq i,j \leq n}$ be a matrix of n^2 variables. For a pair of subsets $I, J \subseteq [n] \underset{\mathrm{def}}{=} \{1, \ldots, n\}$ with $|I| = |J|$, define the (I, J)-minor of x, denoted $\Delta_{I,J}(x)$, to be the determinant of the submatrix of x indexed by rows in I and columns in J. We adopt the convention that the empty minor $\Delta_{\emptyset,\emptyset}(x)$ is equal to 1. An $n \times n$ matrix A is said to be *totally nonnegative* (TNN) if every minor of A is a nonnegative real number. A polynomial $p(x)$ in n^2 variables is called *totally nonnegative* if whenever $A = (a_{i,j})_{1 \leq i,j \leq n}$ is a totally nonnegative matrix, $p(A) \underset{\mathrm{def}}{=} p(a_{1,1}, \ldots, a_{n,n})$ is a nonnegative real number. When taken together, results in [3], [4], [13], [17], [16], and [29] give a graph theoretic characterization of totally nonnegative matrices which is used in [20] to construct several examples of totally nonnegative polynomials.

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Let H denote the infinite array $(h_{j-i})_{i,j\geq 1}$, where h_i denotes the complete homogeneous symmetric function of degree i. (see, for example, [25]) Here we use the convention that $h_i = 0$ whenever $i < 0$. A polynomial $p(x)$ in n^2 variables is called *Schur nonnegative* (SNN) if whenever K is an $n \times n$ submatrix of H, the symmetric function $p(K)$ is a nonnegative linear combination of Schur functions. By the Jacobi identity, the determinant is a trivial example of an SNN polynomial.

Let S_n denote the symmetric group on n letters. For $i \in [n-1]$, let s_i denote the adjacent transposition in S_n which is written $(i, i+1)$ in cycle notation. For a fixed $w \in S_n$, call an expression $s_{i_1} \cdots s_{i_\ell}$ representing w *reduced* if ℓ is minimal. In this case, define the *length* of w, denoted $\ell(w)$, to be ℓ. Let w_o denote the long element of S_n which has one line notation $n(n-1)\dots 1$. Define (*strong*) *Bruhat order* to be the partial order $\leq$ on S_n given by $u \leq v$ if and only if every reduced expression for v contains a subsequence (not necessarily contiguous) which is equal to u. Bruhat order on S_n has the identity permutation e as a unique minimal element, w_o as a unique maximal element, and is a graded poset with rank function given by the length defined above.

For q a formal indeterminate, define the *Hecke algebra* $H_n(q)$ to be the $\mathbb{C}[q^{1/2}, q^{-1/2}]$-algebra with generators $T_{s_1}, \dots, T_{s_{n-1}}$ subject to the relations

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + q, && \text{for } i = 1, \dots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j}, && \text{if } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i}, && \text{if } |i-j| \geq 2. \end{aligned}$$

For $w \in S_n$, define the Hecke algebra element T_w by

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}},$$

where $s_{i_1} \cdots s_{i_\ell}$ is any reduced expression for w. The algebra elements T_w, where w ranges over S_n, form a basis for $H_n(q)$. Specializing at $q = 1$, the map $T_{s_i} \mapsto s_i$ induces an isomorphism between $H_n(1)$ and the symmetric group algebra $\mathbb{C}[S_n]$.

For any $i \in [n-1]$, it is easy to see that the element T_{s_i} is invertible in $H_n(q)$ and that $T_{s_i}^{-1} = \frac{1}{q}(T_{s_i} - q + 1)$. Therefore, any basis element T_w is also invertible in $H_n(q)$ and we can define an involution D of $H_n(q)$ by $D(q^{1/2}) = q^{-1/2}$ and $D(T_w) = T_{w^{-1}}^{-1}$. Under the $q = 1$ specialization which identifies $\mathcal{H}_n(1)$ with $\mathbb{C}[S_n]$, the involution D reduces to the identity map.

The *Kazhdan-Lusztig polynomials* $P_{u,v}(q)$ introduced in [14] can be defined in terms of bases of $H_n(q)$ which are fixed pointwise by the involution D. More specifically, we have the following result.

Lemma 1.1. *There exists a unique family of polynomials* $\{P_{u,v}(q)\}$ *in* $\mathbb{Z}[q]$ *indexed by ordered pairs of permutations* $(u, v) \in S_n^2$ *satisfying the following conditions.*

1. $P_{u,v}(q) = 0$ *unless* $u \leq v$ *in Bruhat order.*
2. The degree of $P_{u,v}(q)$ *is at most equal to* $\frac{\ell(v)-\ell(u)-1}{2}$.
3. $P_{u,u}(q) = 1$ *for any* $u \in S_n$.
4. For any $v \in S_n$, *the element* $C'_v(q)$ *of* $H_n(q)$ *defined by* $C'_v(q) = q^{-\ell(v)/2} \sum_{u \leq v} P_{u,v}(q) T_u$ *is fixed by* D.

The algebra elements

$$C'_v(q) = \sum_{u \leq v} P_{u,v}(q) q^{-\ell(v)/2} T_u, \tag{1.1}$$

which appear in the above Lemma form a basis of $H_n(q)$ called the *Kazhdan-Lusztig basis*. In our present case (type A), we also have that the polynomials $P_{u,v}(q)$ have nonnegative coefficients. With property 2 of the above Lemma in mind, we define a function $\mu : S_n \times S_n \to \mathbb{C}$ by $\mu(w, v) = [q^{\frac{1}{2}(\ell(v)-\ell(w)-1)}] P_{w,v}(q)$. That is, $\mu(w, v)$ is the coefficient of the maximum possible power of q in $P_{w,v}(q)$. Notice that $\mu(w, v) = 0$ whenever $\ell(w) - \ell(v)$ is even.

Recall that a preorder $\leq$ on a set X is a binary relation on X which is transitive and reflexive, but need not be antisymmetric. That is, there may be distinct elements x and y in X satisfying $x \leq y \leq x$. Given X and $\leq$, we have an equivalence relation defined on X via $x \sim y$ if and only if $x \leq y \leq x$. Now $\leq$ induces a partial order $\leq_o$ on the set $X/\sim$ of equivalence classes given by $[x] \leq_o [y]$ if any only if for any elements $x' \in [x]$ and $y' \in [y]$ we have that $x' \leq y'$.

Specializing again to $q = 1$, the elements $\{C'_v(1) \,|\, v \in S_n\}$ form a basis for the symmetric group algebra $\mathbb{C}[S_n]$, which is also called the Kazhdan-Lusztig basis. In [14] this basis is used to define a preorder $\leq_{LR}$ on S_n whose definition we recall here. First define a binary relation $\leq'_{LR}$ on S_n by $u \leq'_{LR} v$ if and only if there exists an $i \in [n-1]$ such that $C'_v(1)$ appears with nonzero coefficient in the expansion of either $s_i C'_u(1)$ or $C'_u(1) s_i$ in the Kazhdan-Lusztig basis of $\mathbb{C}[S_n]$. Let $\leq_{LR}$ be the transitive closure of the relation $\leq'_{LR}$. That is, $u \leq_{LR} v$ if and only if we have a chain $u = w_1 \leq'_{LR} w_2 \leq'_{LR} \cdots \leq'_{LR} w_k = v$. The preorder $\leq_{LR}$ is called the two-sided Kazhdan-Lusztig preorder and it, along with its one-sided analogues, are of great interest in the representation theory of S_n. The equivalence classes on S_n induced by the preorder $\leq_{LR}$ are called two-sided Kazhdan-Lusztig cells.

A polynomial $p(x)$ in n^2 variables is called an *immanant* if it belongs to the $\mathbb{C}$-linear span of $\{x_{1,w(1)} \cdots x_{n,w(n)} \mid w \in S_n\}$. Denote the vector space of immanants by $\mathcal{I}_n(x)$. Following [21], for $w \in S_n$, define the *v-Kazhdan-Lusztig immanant* by

$$\mathrm{Imm}_v(x) \underset{\mathrm{def}}{=} \sum_{w \in S_n} (-1)^{\ell(w)-\ell(v)} P_{w_0 w, w_0 v}(1) x_{1,w(1)} \cdots x_{n,w(n)}. \tag{1.2}$$

In the special case that v is the identity element e of S_n, we have that $\mathrm{Imm}_e(x) = \det(x)$.

Following [26], define the more general *f-immanant* for any function $f : S_n \to \mathbb{C}$ by

$$\mathrm{Imm}_f(x) = \sum_{w \in S_n} f(w) x_{1,w(1)} \cdots x_{n,w(n)}.$$

Typical choices for f include an irreducible character of, or more generally any class function on, the symmetric group S_n.

There exists a certain duality between the Kazhdan-Lusztig basis and the Kazhdan-Lusztig immanants. To state this precisely, for any permutation $v \in S_n$, let $f_v : S_n \to \mathbb{C}$ be the function which defines the v-Kazhdan-Lusztig immanant. That is, $f_v(w) = (-1)^{\ell(w)-\ell(v)} P_{w_0 w, w_0 v}(1)$. We extend f_v to a function $\mathbb{C}[S_n] \to \mathbb{C}$ by linearity. With this definition, we have that

$$f_v(C'_w(1)) = \delta_{v,w}, \tag{1.3}$$

where $C'_w(1)$ is the Kazhdan-Lusztig basis element corresponding to w [21].

It follows from Lemma 1.1 and the fact that the Kazhdan-Lusztig polynomials have nonnegative coefficients in type A that the expression $(-1)^{\ell(w)-\ell(v)} P_{w_0 w, w_0 v}(1)$ is nonzero if and only if $v \leq w$ in the Bruhat order and that $P_{w_0 w, w_0 w}(1) = 1$. Thus, the transition matrix between the set $\{\mathrm{Imm}_w(x) \mid w \in S_n\}$ and the natural basis of immanants $\{x_{1,w(1)} \cdots x_{n,w(n)} \mid w \in S_n\}$ is upper triangular with 1's on the diagonal and the Kazhdan-Lusztig immanants form a basis for the vector space of immanants. The Kazhdan-Lusztig immanants are both TNN and SNN and various examples of TNN and SNN polynomials can be constructed by studying the cone generated by the Kazhdan-Lusztig immanants [21]. Moreover, when w is 321-avoiding, the Kazhdan-Lusztig immanant $\mathrm{Imm}_w(x)$ satisfies a natural generalization of Lindström's Lemma [20].

2. Filtration Equality

To begin, we define two filtrations of the vector space $\mathcal{I}_n(x)$. The first of these is defined using complementary products of matrix minors in the spirit of Désarménian, Kung, and Rota [6] and the second is defined via Kazhdan-Lusztig immanants.

Given a tableau T, write $\mathrm{sh}(T)$ for the shape of the partition corresponding to T. Define the *size* of T to be the integer of which $\mathrm{sh}(T)$ is a partition. T is injective if the numbers $1, 2, \ldots, n$ each appear exactly once in T, where T has size n. T is called *semistandard* if the numbers in T weakly increase across rows and strictly increase down columns, and T is called *standard* if it is both injective and semistandard.

Following Désarménien, Kung, and Rota we define a *bitableau* $(U{:}T)$ to be an ordered pair of tableaux (U, T) such that U and T have the same shape. A bitableau is called injective, semistandard, or standard if both of its entries have the corresponding property. Define the shape of a bitableau $(U{:}T)$, written $\mathrm{sh}(U{:}T)$, to be either $\mathrm{sh}(U)$ or $\mathrm{sh}(T)$. Define the size of $(U{:}T)$ similarly.

Given any bitableau $(U{:}T)$ of size n such that the entries of U and T are drawn from the set $[n]$, we may define an element of the polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{nn}]$ as follows. Suppose that the columns of U and T, viewed as subsets of $[n]$ are $I_1, \ldots, I_k$ and $J_1, \ldots, J_k$, respectively. Then, the product of minors

$$\Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x)$$

is an element of $\mathbb{C}[x_{11}, \ldots, x_{nn}]$. We denote this polynomial by $(U{:}T)(x)$, and think of it as the bitableau $(U{:}T)$ evaluated on the set of variables x. We may also refer to the polynomial $(U{:}T)(x)$ as a bitableau. While it is not in general true that an arbitrary bitableau $(U{:}T)(x)$ with entries drawn from $[n]$ is an immanant on the variable set x, it is easy to see that $(U{:}T)(x)$ is contained in $\mathcal{I}_n(x)$ if and only if $(U{:}T)$ is injective. In this case, the above minor product is a complementary product of minors, i.e., we have that $I_1 \uplus \cdots \uplus I_k = J_1 \uplus \cdots \uplus J_k = [n]$.

Désarménien, Kung, and Rota [6, Thm. p. 68] showed that semistandard bitableaux form a basis of $\mathbb{C}[x_{11}, \ldots, x_{nn}]$. Restricting to standard bitableaux and the subspace $\mathcal{I}_n(x)$, this naturally leads to our first filtration of $\mathcal{I}_n(x)$. Given $k \in \mathbb{N}$, define $\mathcal{U}_{n,k}(x)$ to be the $\mathbb{C}-$linear span of all injective bitableau $(U{:}T)(x)$, where $|(U{:}T)| = n$ and the first part of $\mathrm{sh}(U{:}T)$ is $\leq k$. That is, $\mathcal{U}_{n,k}(x)$ is the span of all complementary products of k (or fewer) minors. By our definition of the empty minor $\Delta_{\emptyset,\emptyset}(x)$, it is clear that

$$\mathcal{U}_{n,1}(x) \subseteq \mathcal{U}_{n,2}(x) \subseteq \cdots \subseteq \mathcal{U}_{n,n}(x) = \mathcal{I}_n(x). \tag{2.1}$$

Thus, the sequence of spaces in (2.1) is a filtration of $\mathcal{I}_n(x)$, which we shall call the *$\mathcal{U}$-filtration.*

In [20], Kazhdan-Lusztig immanants are used to show that the dimension of $\mathcal{U}_{n,2}(x)$ is equal to the n^{th} Catalan number C_n. In this paper we shall relate the dimension of $\mathcal{U}_{n,k}(x)$ for arbitrary k to pattern avoidance in S_n.

For $k \in \mathbb{N}$, let $S_{n,k}$ denote the set of permutations in S_n which do not have a decreasing subsequence of length $k+1$. For example, in one-line notation, $S_{3,2} = \{123, 213, 132, 312, 231\}$. Notice that $S_{n,k} = S_n$ for all $k \geq n$. Define $\mathcal{V}_{n,k}(x)$ to be the linear span of all Kazhdan-Lusztig immanants $\mathrm{Imm}_w(x)$ corresponding to permutations $w \in S_{n,k}$. The obvious chain of inclusions $S_{n,1} \subseteq S_{n,2} \subseteq \cdots \subseteq S_{n,n}$ gives rise to another filtration of $\mathcal{I}_n(x)$ given by $\mathcal{V}_{n,1}(x) \subseteq \mathcal{V}_{n,2}(x) \subseteq \cdots \subseteq \mathcal{V}_{n,n}(x) \subseteq \mathcal{I}_n(x)$. Call this filtration the $\mathcal{V}$ filtration.

Recall that the Robinson-Schensted correspondence gives an algorithmic bijection between S_n and the set of ordered pairs of standard Young tableaux with n boxes having the same shape. The details of this algorithm can be found, for example, in [22]. In this paper we will be using *column* insertion only, so that the long element $w_0 \in S_n$ will correspond to $(12\cdots n, 12\cdots n)$. In order to prove the equality of the $\mathcal{U}$ and $\mathcal{V}$ filtrations, let us first examine the image of $S_{n,k}$ under the Robinson-Schensted correspondence. Let $s_{[1,k]}$ be the longest element in the subgroup of S_n generated by $s_1, \ldots, s_{k-1}$.

Lemma 2.1. *Suppose $v \notin S_{n,k-1}$. Then we have $v \leq_{LR} s_{[1,k]}$.*

Proof. Given any permutation w, define the pair of tableaux $(P'(w), Q'(w))$ to be the image of w under the Robinson-Schensted column insertion correspondence. Let $\lambda'(w)$ be the shape of these tableaux.

A well-known property of the Robinson-Schensted correspondence implies that $\lambda'(v) \geq \lambda'(s_{[1,k]})$ in the dominance order. This dominance relation in turn is known to be equivalent to the partial order on Kazhdan-Lusztig cells induced by the preorder $\leq_{LR}$. Thus in the preorder $\leq_{LR}$, every permutation in the cell of v precedes every permutation in the cell of $s_{[1,k]}$. (See [1], [9, Sec. 1], [12, Appendix].) □

Our first main result is a generalization of the fact that the determinant vanishes on matrices having two equal rows. This also generalizes [20, Prop. 3.14], which together with [21] implies that Proposition 2.2 holds when $k = 2$.

Proposition 2.2. *Suppose $A \in \mathrm{Mat}_n(\mathbb{C})$ has k equal rows and let $v \in S_{n,k-1}$. Then,* $\mathrm{Imm}_v(A) = 0$.

Proof. As in [27], define the element $[A]$ of $\mathbb{C}[S_n]$ by

$$[A] = \sum_{w \in S_n} a_{1,w(1)} \cdots a_{n,w(n)} w.$$

Let $i_1 < \cdots < i_k$ be the indices of k rows in A which are equal and let U be the subgroup of S_n which fixes all indices not contained in the set $\{i_1, \ldots, i_k\}$. Then

$$\sum_{u \in U} u$$

factors as $wz_{[1,k]}w'$ for some elements w, w' of S_n. Since every element $w \in S_n$ factors as $w = uv$ for some $u \in U$ and v in an appropriate set of coset representatives, it follows that $[A]$ factors as

$$\begin{aligned}[A] &= \left(\sum_{u \in U} u\right) g(A) \\ &= (wz_{[1,k]}w')g(A)\end{aligned}$$

for some group algebra element $g(A)$.

Let I be the linear span of $\{C'_u(1) \,|\, u \leq_{LR} s_{[1,k]}\}$ in $\mathbb{C}[S_n]$. It follows from properties of the preorder $\leq_{LR}$ that I is in fact a two-sided ideal in $\mathbb{C}[S_n]$ and the set $\{C'_u(1) \,|, u \leq_{LR} s_{[1,k]}\}$ is a basis for this ideal. Let $\theta : \mathbb{C}[S_n] \to \mathbb{C}[S_n]/I$ be the canonical homomorphism. Since $z_{[1,k]} = C'_{s_{[1,k]}}(1)$ belongs to I, we have $\theta([A]) = 0$.

On the other hand, by the duality of Kazhdan-Lusztig immanants and the Kazhdan-Lusztig basis [21, Eq. 4] we have that

$$\begin{aligned}\theta([A]) &= \theta\left(\sum_{w \in S_n} \mathrm{Imm}_w(A) C'_w(1)\right) \\ &= \sum_{w \in S_n} \mathrm{Imm}_w(A)\theta(C'_w(1)).\end{aligned}$$

Since $\theta(C'_w(1)) = 0$ for all permutations $w \leq_{LR} s_{[1,k]}$, we have

$$0 = \sum_{w} \mathrm{Imm}_w(A)\theta(C'_w(1)),$$

where the sum is over all permutations $w \not\leq_{LR} s_{[1,k]}$, i.e., those permutations having no decreasing subsequence of length k. Since the elements $\theta(C'_w(1))$ in this sum are linearly independent, we must have $\mathrm{Imm}_w(A) = 0$ for each permutation w having no decreasing subsequence of length k. □

It should be noted that the obvious basis free analogue of the previous proposition fails in general. That is, if a complex $n \times n$ complex matrix A has a set of m rows with rank $\leq m - k$ and $w \in S_{n,k}$, it is not necessarily the case that $\mathrm{Imm}_w(A) = 0$. This is because, unlike the determinant, Kazhdan-Lusztig immanants corresponding to permutations other than 1 are not in general independent of basis, as can be readily checked.

On the other hand, by [21] we have that $\mathrm{Imm}_{w^{-1}}(A) = \mathrm{Imm}_w(A^T)$ for any permutation w and matrix A. Here A^T denotes the transpose of the matrix A. Since $S_{n,k}$ is closed under taking inverses of permutations, it follows that the previous proposition remains true when the word 'rows' is replaced by the word 'columns'.

Using Proposition 2.2 we now seek to establish a relation between the $\mathcal{U}$ filtration and the $\mathcal{V}$ filtration.

Proposition 2.3. *Suppose $(U{:}T)(x)$ is a generator of $\mathcal{U}_{n,k}(x)$. Then, there exist numbers $d_w \in \mathbb{C}$ such that $(U{:}T)(x) = \sum_{w \in S_{n,k}} d_w \mathrm{Imm}_w(x)$.*

Proof. Let $I_1, \dots, I_k$ and $J_1, \dots, J_k$ be the column sets of U and T, respectively.

The Kazhdan-Lusztig immanants form a basis for the vector space of immanants, so we may write

$$\Delta_{I_1,J_1}(x)\cdots\Delta_{I_k,J_k}(x) = \sum_{w\in S_n} d_w \mathrm{Imm}_w(x), \tag{2.2}$$

for some numbers $d_w \in \mathbb{C}$. We show that $d_w = 0$ whenever $w \notin S_{n,k}$.

Suppose that in Equation (2.2) we have $d_w \neq 0$ for some permutation $w \notin S_{n,k}$. Let m be the greatest index for which such a permutation belongs to $S_{n,m}$, and among such elements of $S_{n,m} \smallsetminus S_{n,m-1}$, let y be a Bruhat minimal element. Then, we may rewrite Equation (2.2) as

$$\Delta_{I_1,J_1}(x)\cdots\Delta_{I_k,J_k}(x) = \sum_{w\in S_{n,m-1}} d_w \mathrm{Imm}_w(x) + \sum_{\substack{w\in S_n \\ w\not\leq y}} d_w \mathrm{Imm}_w(x) + d_y \mathrm{Imm}_y(x). \tag{2.3}$$

By the definition of $S_{n,m}$ we may choose indices $i_1 < \cdots < i_m$ such that $y(i_1) > \cdots > y(i_m)$. Let $D \in \mathrm{Mat}_n(\mathbb{C})$ be the matrix obtained by replacing all entries in the rows $i_1, \ldots, i_m$ of the permutation matrix of y with ones. Since D has $m \geq k+1$ equal rows, the pigeonhole principle implies that some pair of these rows have indices contained in one of the sets $I_1, \ldots, I_k$. Hence, $\Delta_{I_1,J_1}(D)\cdots\Delta_{I_k,J_k}(D) = 0$.

By Proposition 2.2, we have $\mathrm{Imm}_w(D) = 0$ for every $w \in S_{n,m}$, and by Equation (1.2), we have $\mathrm{Imm}_w(D) = 0$ for every $w \not\leq y$ in the Bruhat order. Furthermore, it is easy to see that $\mathrm{Imm}_y(D) = 1$. Thus, applying both sides of Equation (2.3) to D, we obtain $0 = d_y$, a contradiction. We conclude that $d_w = 0$ for all $w \in S_{n,m} \smallsetminus S_{n,m-1}$ whenever $m > k$, as desired. □

Properties of the dual canonical basis of $\mathcal{O}(SL_n(\mathbb{C}))$ imply that the coefficients d_w in Proposition 2.3 are in fact nonnegative integers. In the special case $k = 2$, results in [21], [20] give a combinatorial proof of this nonnegativity. For k arbitrary and in the special case that w avoids the patterns 3412 and 4231 (i.e., when the Schubert variety Γ_w corresponding to w is smooth), results in [24] give another proof.

The equality of the $\mathcal{U}$ and $\mathcal{V}$ filtrations now follows rather easily from Proposition 2.3.

Theorem 2.4. *The $\mathcal{U}$ and $\mathcal{V}$ filtrations of $\mathcal{I}_n(x)$ are equal. That is, $\mathcal{U}_{n,k}(x) = \mathcal{V}_{n,k}(x)$ for all n and k.*

Proof. Proposition 2.3 implies that $\mathcal{U}_{n,k}(x) \subseteq \mathcal{V}_{n,k}(x)$ and the linear independence of the Kazhdan-Lusztig immanants implies that $\dim \mathcal{V}_{n,k}(x) = |S_{n,k}|$. Recall that the RSK correspondence implies that $|S_{n,k}|$ is also equal to the number of pairs (U, T) of tableaux of shape λ with $\lambda \vdash n$ and $\lambda_1 \leq k$, and that the corresponding bitableaux span $\mathcal{U}_{n,k}(x)$. Thus we have the desired equality. □

With this result in hand, we henceforth denote either of the spaces $\mathcal{U}_{n,k}(x)$ or $\mathcal{V}_{n,k}(x)$ by $\mathcal{I}_{n,k}(x)$. It may be interesting to note that the irreducible character immanants, usually denoted $\mathrm{Imm}_\lambda(x)$ in the literature (e.g., [27]), fit very nicely into our filtration. Using [19, p. 238], one sees that $\mathrm{Imm}_\lambda(x)$ belongs to the set difference $\mathcal{I}_{n,\lambda_1}(x) \smallsetminus \mathcal{I}_{n,\lambda_1-1}(x)$.

The numbers $|S_{n,k}|$ were studied by Gessel [11] who found an expression involving Bessel functions for the generating function $\sum_{n\geq 1} |S_{n,k}|t^n$. The authors do not know of a simple form of the polynomial $\sum_{k=1}^n |S_{n,k}|t^k$. Désarménien [5] has given combinatorial interpretations for the transition matrix relating the basis of standard bitableaux to the natural basis $\{x_{1,v(1)} \cdots x_{n,v(n)} \,|\, v \in S_n\}$. (See Stokke [28] for a quantum version of this result.) It would also be interesting to investigate the transition matrix between the bases of standard bitableaux and Kazhdan-Lusztig immanants.

Combining Theorem 2.4 with the characterization of the dual canonical basis in [24], we may easily extend our results to obtain information about the full polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{nn}]$. Specifically, given any $m \in \mathbb{N}$, define an $m \times m$ *generalized submatrix* of the $n \times n$ matrix x to be any matrix of the form

$$(x_{a(i),b(j)})_{1\leq i,j\leq m},$$

where $1 \leq a(1) \leq \cdots \leq a(m) \leq n$ and $1 \leq b(1) \leq \cdots \leq b(m) \leq n$. Define the set $\Gamma_{n,m,k}(x)$ by $\Gamma_{n,m,k}(x) = \{\mathrm{Imm}_w(y) \,|\, m \in \mathbb{N}, w \in S_{m,k}\}$, where y ranges over all $m \times m$ generalized submatrices of x. It has been shown in [9] and [24] that the nonzero elements of the union $\bigcup_{m,k\geq 0} \Gamma_{n,m,k}(x)$ (modulo $\det(x) - 1$) are precisely the dual canonical basis elements of the coordinate ring $\mathcal{O}(SL_n(\mathbb{C}))$. In analogy with our definition of $\mathcal{V}_{n,k}(x)$, define $\mathcal{V}'_{n,m,k}(x) = \mathrm{span}_{\mathbb{C}}(\Gamma_{n,m,k}(x))$. In analogy with the $\mathcal{U}$ filtration, define $\mathcal{U}'_{n,m,k}(x)$ to be the span of all semistandard bitableau $(U{:}T)(x)$ of size m and such that U and T have entries in $[n]$. By specializing Theorem 2.4 to the case where some rows and columns of x are equal, we get the following.

Corollary 2.5. *For all positive integers n, m, k we have that $\mathcal{U}'_{n,m,k}(x) = \mathcal{V}'_{n,m,k}(x)$.*

3. Products of Immanants

Recalling that the determinant is the Kazhdan-Lusztig immanant corresponding to the identity permutation, we see that the problem of expanding bitableaux on x in the basis of Kazhdan-Lusztig immanants $\{\mathrm{Imm}_w(x) | w \in S_n\}$ is a problem of multiplying together certain Kazhdan-Lusztig immanants of submatrices of x and expanding the result in the Kazhdan-Lusztig immanants basis. In this section we consider the more general situation of analyzing these expansions where the immanants in the product do not all necessarily correspond to the permutation 1.

Given an $n \times n$ matrix $x = (x_{i,j})$ and subsets I, J of $[n]$, define the I, J *submatrix* of x to be

$$x_{I,J} \underset{\mathrm{def}}{=} (x_{i,j})_{i \in I, j \in J}.$$

Assuming that $|I| = |J|$ and defining $\overline{I} = [n] \smallsetminus I$, $\overline{J} = [n] \smallsetminus J$, one sees immediately that any product of immanants of $x_{I,J}$ and $x_{\overline{I},\overline{J}}$ is an immanant of x. Moreover, one may use properties of the dual canonical basis of to show that a product of Kazhdan-Lusztig immanants of such submatrices expands with nonnegative coefficients in the Kazhdan-Lusztig immanant basis of $\mathcal{I}_n(x)$. Combinatorial interpretations of these coefficients have been given in [21], [20] when the two immanants are minors. These results (or alternately Theorem 2.4) show that a product of two complementary minors belongs to $\mathcal{I}_{n,2}(x)$. More generally, we have the following result which states that in the expansion of a product $\mathrm{Imm}_u(x_{I,J})\mathrm{Imm}_v(x_{\overline{I},\overline{J}})$ in terms of the Kazhdan-Lusztig immanant basis of $\mathcal{I}_n(x)$, the immanants appearing with nonzero coefficient are indexed by permutations whose longest decreasing subsequences are bounded in terms of u and v.

Corollary 3.1. *Given index sets I, J with $|I| = |J| = k$ and permutations $u \in S_{k,a}$, $v \in S_{n-k,b}$, then the product $\mathrm{Imm}_u(x_{I,J})\mathrm{Imm}_v(x_{\overline{I},\overline{J}})$ belongs to $\mathcal{I}_{n,a+b}(x)$.*

Proof. Since $\mathrm{Imm}_u(x_{I,J})$ belongs to $\mathcal{I}_{k,a}(x_{I,J})$, it is equal to a linear combination of products of at most a minors of $x_{I,J}$. Similarly, Imm_v is equal to a linear combination of at most b minors of $x_{\overline{I},\overline{J}}$. By definition, the product of these linear combinations belongs to $\mathcal{I}_{n,a+b}(x)$. □

Note that a direct proof of Corollary 3.1 in terms of the $\mathcal{V}$ filtration would have involved the identification of various sums of products of Kazhdan-Lusztig polynomials, while Theorem 2.5 enables us to give a very simple proof in terms of the $\mathcal{U}$ filtration.

No simple formula is known for the expansion of a general product of the form $\mathrm{Imm}_u(x_{I,J})\mathrm{Imm}_v(x_{\overline{I},\overline{J}})$ in terms of the Kazhdan-Lusztig immanant basis of $\mathcal{I}_n(x)$. However, the following result gives such an expansion in the special case that the submatrices $x_{I,J}$ and $x_{\overline{I},\overline{J}}$ are related antidiagonally within x. That is,

$$\begin{aligned} I &= [k] & J &= \{n-k+1, \dots, n\} \\ \overline{I} &= \{k+1, \dots, n\} & \overline{J} &= [n-k]. \end{aligned} \tag{3.1}$$

Theorem 3.2. *The Kazhdan-Lusztig immanant $\mathrm{Imm}_w(x)$ factors as a product of Kazhdan-Lusztig immanants of submatrices of x if and only if there exists an index $k < n$ such that $\{w(k+1), \dots, w(n)\} \subseteq [k]$. In this case we have*

$$\mathrm{Imm}_w(x) = \mathrm{Imm}_u(x_{I,J})\mathrm{Imm}_v(x_{\overline{I},\overline{J}}),$$

where I, $\overline{I}$, J, $\overline{J}$ are defined by (3.1) and $u \in S_k$, $v \in S_{n-k}$ are defined in terms of the longest elements w_0, w_0', w_0'' of S_n, S_k, S_{n-k} by $ww_0 = uw_0' \oplus vw_0''$.

Proof. To economize notation, we shall write $\epsilon_{w,w'} = (-1)^{\ell(w')-\ell(w)}$ and $Q_{w,w'} = P_{w_0w,w_0w'} = P_{ww_0,w'w_0}$, for permutations $w, w' \in S_m$ and the corresponding longest element $w_0 \in S_m$. Suppose that there exist $u \in S_k$, $v \in S_{n-k}$ satisfying $ww_0 = uw_0' \oplus vw_0''$. Then we have

$$\text{Imm}_w(x) = \sum_{t \geq (uw_0' \oplus vw_0'')w_0} \epsilon_{w,t} Q_{w,t}(1) x_{1,t(1)} \cdots x_{n,t(n)}. \tag{3.2}$$

Note that t satisfies $t \geq (uw_0' \oplus vw_0'')w_0$ if and only if we have $t = (yw_0' \oplus zw_0'')w_0$ for $y \geq u$ and $z \geq v$. In this case, the one-line notation for t is

$$(n-k+y(1)) \cdots (n-k+y(k)) \cdot z(1) \cdots z(n-k)$$

and we have

$$\begin{aligned} \epsilon_{w,t} &= \epsilon_{u,y}\epsilon_{v,z}, \\ Q_{w,t}(1) &= Q_{(uw_0' \oplus vw_0'')w_0,(yw_0' \oplus zw_0'')w_0}(1) \\ &= P_{(yw_0' \oplus zw_0''),(uw_0' \oplus vw_0'')}(1) \\ &= P_{yw_0',uw_0'}(1) P_{zw_0'',vw_0''}(1) \\ &= Q_{u,y}(1) Q_{v,z}(1). \end{aligned}$$

Thus, Equation (3.2) becomes

$$\begin{aligned} \text{Imm}_w(x) &= \sum_{\substack{y \geq u \\ z \geq v}} \epsilon_{u,y}\epsilon_{v,z} Q_{u,y}(1) Q_{v,z}(1) x_{1,n-k+y(1)} \cdots x_{k,n-k+y(n)} x_{k+1,z(1)} \cdots x_{n,z(n-k)} \\ &= \text{Imm}_u(x_{I,J}) \text{Imm}_v(x_{\overline{I},\overline{J}}), \end{aligned}$$

where I, $\overline{I}$, J, and $\overline{J}$ are as in (3.1).

Now suppose that $\text{Imm}_w(x)$ factors as a product of Kazhdan-Lusztig immanants of submatrices of x

$$\text{Imm}_w(x) = \text{Imm}_u(x_{I,J}) \text{Imm}_v(x_{\overline{I},\overline{J}}) \tag{3.3}$$

in at least one way, but that for no such factorization do the permutations u, v satisfy the required identity. It follows that the sets I, $\overline{I}$, J, $\overline{J}$ do not satisfy (3.1). Assume that we have named the index sets in all factorizations (3.3) so that we have $1 \in I$.

Choose a particular factorization and let m be the smallest element of $\overline{I}$. Suppose that there exists an index $i \in I$ such that $i > m$ and $w(i) > w(m)$. Transposing the letters in the ith and mth positions of w, we obtain a permutation greater than w in the Bruhat order. Thus the corresponding monomial

$$x_{1,w(1)} \cdots x_{i,w(m)} \cdots x_{m,w(i)} \cdots x_{n,w(n)}$$

appears with nonzero coefficient in $\mathrm{Imm}_w(x)$. Observe however that this monomial does not appear in the product $\mathrm{Imm}_u(x_{I,J})\mathrm{Imm}_v(x_{\overline{I},\overline{J}})$ because the variables $x_{i,w(m)}$ and $x_{m,w(i)}$ appear in neither of the submatrices $x_{I,J}$, $x_{\overline{I},\overline{J}}$. Thus the product of immanants is not equal to $\mathrm{Imm}_w(x)$, and we deduce that $m = k+1$ and that $I = [m-1]$.

Now we claim that the sets I, J, $\overline{I}$, and $\overline{J}$ satisfy the equations $J = \{n+1-i \mid i \in I\}$ and $\overline{J} = \{n+1-i' \mid i' \in \overline{I}\}$. If this is not the case, then the monomial $x_{1,n} \cdots x_{1,n}$ corresponding to w_0 appears with coefficient zero on the right-hand side of (3.3), and with coefficient ± 1 on the left-hand side, a contradiction. From this claim it follows immediately that the sets I, $\overline{I}$, J, $\overline{J}$ satisfy (3.1) and that the permutations u, v satisfy $ww_0 = (uw_0' \oplus vw_0'')$.

We conclude that the existence of any factorization of the form (3.3) implies the existence of one which satisfies the conditions of the theorem. □

To illustrate the theorem with an example, let us factor $\mathrm{Imm}_{365421}(x)$. Writing

$$(365421)w_0 = 412365 = 4123 \oplus 21 = (1432)w_0' \oplus (12)w_0'',$$

where w_0' and w_0'' are the longest elements in S_4 and S_2, respectively, we have that $\mathrm{Imm}_{365412}(x) = \mathrm{Imm}_{1432}(x_{1234,3456})\mathrm{Imm}_{12}(x_{56,12})$.

In the event that $x_{I,J}$ and $x_{\overline{I},\overline{J}}$ are not antidiagonally related within x, the expansion of $\mathrm{Imm}_u(x_{I,J})\mathrm{Imm}_v(x_{\overline{I},\overline{J}})$ in the Kazhdan-Lusztig immanant basis of $\mathcal{I}_n(x)$ is in general more delicate. It is easy to see that this expansion has the form

$$\mathrm{Imm}_u(x_{I,J})\mathrm{Imm}_v(x_{\overline{I},\overline{J}}) = \sum_{y \geq w} d_y \mathrm{Imm}_y(x)$$

where w is permutation whose matrix P has submatrices $P_{I,J}$ and $P_{\overline{I},\overline{J}}$ equal to the permutation matrices of u and v. The problem of determining the coefficients d_y can in principle be solved using [21, Prop. 6.3]. Specifically, for each $i \in [n-1]$ let P_i be the permutation matrix of the adjacent transposition s_i. For $w \in S_n$, the above result states that

$$(3.4) \quad \begin{aligned} \mathrm{Imm}_w(Px) &= \begin{cases} -\mathrm{Imm}_w(x) & \text{if } sw > w, \\ \mathrm{Imm}_w(x) + \mathrm{Imm}_{sw}(x) + \sum_{sz>z} \mu(w,z)\mathrm{Imm}_z(x) & \text{if } sw < w \end{cases} \\ \mathrm{Imm}_w(xP) &= \begin{cases} -\mathrm{Imm}_w(x) & \text{if } ws > w, \\ \mathrm{Imm}_w(x) + \mathrm{Imm}_{ws}(x) + \sum_{zs>z} \mu(w,z)\mathrm{Imm}_z(x) & \text{if } ws < w. \end{cases} \end{aligned}$$

It is clear that some pair of sequences $P_{i_1}, \ldots, P_{i_k}$, $P_{j_1}, \ldots P_{j_\ell}$ of the above form have the property that the submatries corresponding to $x_{I,J}$ and $x_{\overline{I},\overline{J}}$ in $P_{i_1} \cdots P_{i_k} x P_{j_1} \cdots P_{j_\ell}$ are in block antidiagonal position. Therefore, the above equation may be used to inductively determine the expansion of $\mathrm{Imm}_u(y)\mathrm{Imm}_v(z)$ in the basis of Kazhdan-Lusztig immanants of x itself. The efficiency of this method is bounded by our ability to compute the μ function.

The above equation also has application to the $0, 1-$conjecture. It had been suspected that the $\mu(w, z)$ was equal to either 0 or 1 for any permutations $w, z \in S_n$. This conjecture was disproven by McLarnan and Warrington [18], but the above equation implies that this conjecture is true in some cases.

Proposition 3.3. *Let w and z be permutations in S_n and suppose that w is contained in $S_{n,2}$. Suppose also that there exists a simple transposition s_i such that either $s_iw < w$ and $s_iz > z$ or $ws_i < w$ and $zs_i > z$. Then, $\mu(w, z)$ is equal to either 0 or 1.*

Proof. Combine Proposition 3.12 of [20], the equivalence of the Kazhdan-Lusztig immanants corresponding to permutations in $S_{n,2}$ and Temperley-Lieb immanants proven in [21], and the linear independence of the Kazhdan-Luszing immanants. □

4. Total nonnegativity and Schur nonnegativity

The problem of deciding the total nonnegativity or Schur nonnegativity of an immanant is not easy. In particular, there is no known algorithm to do this, unless we restrict our attention to $\mathcal{I}_{n,2}$ [20, Thm. 4.5]. Nevertheless, it is possible to state some simple sufficient conditions which apply to immanants which are differences of products of minors.

Define the poset $P_{n,k}$ on products of (at most) k complementary minors by

$$\Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x) \leq \Delta_{I'_1,J'_1}(x) \cdots \Delta_{I'_k,J'_k}(x)$$

if and only if the difference $\Delta_{I'_1,J'_1}(x) \cdots \Delta_{I'_k,J'_k}(x) - \Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x)$ is TNN.

In [23, Thm. 3.2] and [20, Prop. 4.1, Thm. 4.2, Cor. 4.6], the authors give several simple combinatorial characterizations of $P_{n,2}$. These characterizations imply that this poset has a unique maximal element $\Delta_{I,I}(x)\Delta_{J,J}(x)$ given by $I = \{1, 3, 5, \ldots, \}$, $J = \{2, 4, 6, \ldots, \}$ and that the determinant $\Delta_{[n],[n]}(x)\Delta_{\emptyset,\emptyset}(x)$ is among the n minimal elements. In [21] the authors show that the combinatorial tests in [20] provide sufficient conditions for an immanant in $\mathcal{I}_{n,2}(x)$ to be SNN. Therefore, whenever $\Delta_{I,J}(x)\Delta_{I',J'}(x) \leq \Delta_{K,L}(x)\Delta_{K',L'}(x)$ in $P_{n,2}$ we also have that $\Delta_{K,L}(x)\Delta_{K',L'}(x) - \Delta_{I,J}(x)\Delta_{I',J'}(x)$ is SNN. It is unknown whether the converse of this statement is true.

In [7], [8], and [21], Drake, Gerrish, and the authors study the poset $P_{n,n} \smallsetminus P_{n,n-1}$ of products of n nonempty minors, that is, permutation monomials $x_{1,w(1)} \cdots x_{n,w(n)}$,

$w \in S_n$. This poset is isomorphic to (the dual of) the Bruhat order, with unique maximal element $x_{1,1} \cdots x_{n,n}$, and unique minimal element $x_{1,n} \cdots x_{n,1}$. The comparison $x_{1,w(1)} \cdots x_{n,w(n)} \leq x_{1,v(1)} \cdots x_{n,v(n)}$ is equivalent to each of the following statements.

(1) The difference $x_{1,v(1)} \cdots x_{n,v(n)} - x_{1,w(1)} \cdots x_{n,w(n)}$ is TNN.
(2) The difference $x_{1,v(1)} \cdots x_{n,v(n)} - x_{1,w(1)} \cdots x_{n,w(n)}$ is SNN.
(3) The difference $x_{1,v(1)} \cdots x_{n,v(n)} - x_{1,w(1)} \cdots x_{n,w(n)}$ is a nonnegative linear combination of Kazhdan-Lusztig immanants.
(4) $v \leq w$ in the Bruhat order.

In analogy to some of the above results we show that $P_{n,k}$ has a unique maximal element for arbitrary k, and that certain comparable elements of $P_{n,k}$ have differences which are SNN as well as TNN.

Lemma 4.1. *Let $(I_1, \ldots, I_p)$ and $(J_1, \ldots, J_p)$ be sequences of sets satisfying $|I_i| = |J_i|$ for all i, and $I_1 \uplus \cdots \uplus I_p = J_1 \uplus \cdots \uplus J_p = [n]$. Fix indices $k \leq \ell$ and let $\alpha_1 < \cdots < \alpha_p$ be the elements of $I_k \cup I_\ell$, and $\beta_1 < \cdots < \beta_p$ be the elements of $J_k \cup J_\ell$. Define two more sequences of sets $(I'_1, \ldots, I'_p)$ and $(J'_1, \ldots, J'_p)$ by*

$$I'_i = \begin{cases} \{\alpha_1, \alpha_3, \ldots\} & \text{if } i = k, \\ \{\alpha_2, \alpha_4, \ldots\} & \text{if } i = \ell, \\ I_i & \text{otherwise}, \end{cases} \qquad J'_i = \begin{cases} \{\alpha'_1, \alpha'_3, \ldots\} & \text{if } i = k, \\ \{\alpha'_2, \alpha'_4, \ldots\} & \text{if } i = \ell, \\ J_i & \text{otherwise}. \end{cases}$$

Then the immanant $\Delta_{I'_1,J'_1}(x) \cdots \Delta_{I'_k,J'_k}(x) - \Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x)$ is totally nonnegative and Schur nonnegative.

Proof. This difference is

$$\frac{\Delta_{I'_1,J'_1}(x) \cdots \Delta_{I'_p,J'_p}(x)}{\Delta_{I'_k,J'_k}(x)\Delta_{I'_\ell,J'_\ell}(x)}(\Delta_{I'_k,J'_k}(x)\Delta_{I'_\ell,J'_\ell}(x) - \Delta_{I_k,J_k}(x)\Delta_{I_\ell,J_\ell}(x)),$$

which is TNN and SNN by [21, Thm. 5.2] and [20, Prop. 4.6]. □

Like $P_{n,2}$ and $P_{n,n} \smallsetminus P_{n,n-1}$, each poset $P_{n,k}$ has a unique maximal element.

Theorem 4.2. *Let $(I_1, \ldots, I_p)$ and $(J_1, \ldots, J_p)$ be sequences of sets as in Lemma 4.1, and define a third sequence $(K_1, \ldots, K_p)$ by*

$$K_j = \{i \in [n] \,|\, i \equiv j \pmod p\}.$$

Then the immanant $\Delta_{K_1,K_1}(x) \cdots \Delta_{K_p,K_p}(x) - \Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x)$ is totally nonnegative and Schur nonnegative.

Proof. Applying several iterations of Lemma 4.1 to the sets $I_1, \ldots, I_p, J_1, \ldots, J_p$, we obtain the desired result. □

This theorem yields an easy method of constructing families of TNN and SNN polynomials.

Corollary 4.3. *Let* $k \leq \ell$ *and define the sequences of sets* $(I_1, \ldots, I_k)$, $(J_1, \ldots, J_\ell)$ *by* $I_j = \{i \in [n] \,|\, i \equiv j \pmod{k}\}$, $J_j = \{i \in [n] \,|\, i \equiv j \pmod{\ell}\}$. *Then the immanant* $\Delta_{J_1,J_1}(x) \cdots \Delta_{J_\ell,J_\ell}(x) - \Delta_{I_1,I_1}(x) \cdots \Delta_{I_k,I_k}(x)$ *is totally nonnegative and Schur nonnegative.*

For example, we may apply the immanant

$$\Delta_{14,14}(x)\Delta_{25,25}(x)\Delta_{3,3}(x) - \Delta_{135,135}(x)\Delta_{24,24}(x)$$

to the Jacobi-Trudi matrix

$$\begin{bmatrix} h_9 & h_{10} & h_{11} & h_{12} & h_{13} \\ h_6 & h_7 & h_8 & h_9 & h_{10} \\ h_4 & h_5 & h_6 & h_7 & h_8 \\ h_3 & h_4 & h_5 & h_6 & h_7 \\ 1 & h_1 & h_2 & h_3 & h_4 \end{bmatrix}$$

to deduce that the symmetric function

$$s_{(11,6)/2}s_{(9,4)/2}s_6 - s_{(11,7,4)/(2,1)}s_{(8,6)/1}$$

is SNN.

Not much is known about the posets $P_{n,k}$ in general. Obviously we have that $P_{n,1} \subset P_{n,2} \subset \cdots \subset P_{n,n}$. By Theorem 2.6 $P_{n,n}$ contains a subposet isomorphic to (the dual of) the Bruhat order on S_n. Also, it is possible to show that any element of $\mathcal{I}_3(x)$ is TNN or SNN if and only if it may be expressed as a nonnegative linear combination of Kazhdan-Lusztig immanants. In particular, this allows one to construct the poset $P_{3,3}$ and see that it coincides with the analogous poset constructed by considering SNN differences. Boocher and Froehle [2] have produced several conjectures concerning the poset $P_{n,3}$ with numerical evidence for the $n = 4$ case. It would be interesting to see what $P_{n,k}$ looks like in general.

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