The Final Exam

The final exam will **not** include material from Set Theory (from pages 1-8 of AC), or sentential logic.

The Gödel Completeness Theorem

Theorem (Extended Gödel Completeness Theorem)

Let Γ *be a consistent set of wffs of a first-order language* $\mathbb{L}$*. There is a structure* $\mathfrak{A}$ *for* $\mathbb{L}$ *such that:*

- Γ *is satisfiable in* $\mathfrak{A}$*, and*
- $\overline{\overline{|\mathfrak{A}|}} \leq \bar{\bar{\mathbb{L}}}$.

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- *$\overline{\overline{|\mathfrak{A}|}} \leq \bar{\bar{\mathbb{L}}}$.*

Theorem (Gödel Completeness Theorem)

For every set Γ of wffs of a first-order language $\mathbb{L}$, and wff φ,

$$\Gamma \models \varphi \implies \Gamma \vdash \varphi \, .$$

Some Consequences of the Gödel Completeness Theorem

Since we already have shown Soundness, we have:

Theorem

- $\Gamma \models \varphi$ *iff* $\Gamma \vdash \varphi$.
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Γ *is consistent* *iff* Γ *is satisfiable.*

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Theorem

If $\Gamma \models \varphi$ then there is a finite subset Δ of Γ such that $\Delta \models \varphi$.

The Downwards Löwenheim-Skolem Theorem

Theorem (DLST)

If Γ *is satisfiable then* Γ *is satisfiable in a structure* $\mathfrak{A}$ *with* $\overline{\overline{|\mathfrak{A}|}} \leq \bar{\bar{\mathbb{L}}}$.

In particular, if Γ *is a satisfiable set of sentences of an enumerable language* $\mathbb{L}$ *then* Γ *is satisfiable in a countable structure.*

Completeness Theorem and the Effective Enumerability of Valid Wffs

Theorem (Informal)

If the language $\mathbb{L}$ *is fairly simple, for example if there are only finitely many relation symbols, function symbols, and constant symbols, then the set of valid wffs of* $\mathbb{L}$ *is effectively enumerable.*

Proof of the Extended Gödel Completeness Theorem

Let $\mathbb{L}$ be a first-order language. If $\doteq$ is in $\mathbb{L}$, let $\mathbb{L}'$ be the language obtained by taking $\doteq$ out of $\mathbb{L}$ and replacing it by a new binary relation symbol $\dot{E}$. If $\doteq$ is not in $\mathbb{L}$ let $\mathbb{L}'$ be the same as $\mathbb{L}$.

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(Note that the terms of $\mathbb{L}'$ are the same as the terms of $\mathbb{L}$.)

For each wff φ of $\mathbb{L}$ let φ^* be the wff of $\mathbb{L}'$ obtained by replacing in φ each occurrence of $\doteq$ by $\dot{E}$. For Δ a set of wffs of $\mathbb{L}$ let $\Delta^* = \{\varphi^* \mid \varphi \in \Delta\}$.

Lemma 1

Lemma (1)

Let Δ *be a set of wffs of* $\mathbb{L}$ *such that:*

1. Δ *is consistent;*
2. *for every* φ *of* $\mathbb{L}$, $\quad \varphi \in \Delta$ *or* $(\neg\varphi) \in \Delta$*;*

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2. *for every φ of $\mathbb{L}$,* $\varphi \in \Delta$ *or* $(\neg\varphi) \in \Delta$*;*
3. *for every wff φ of $\mathbb{L}$ and every variable x, there is a constant symbol c such that* $(\neg\dot{\forall} x\, \varphi \rightarrow \neg\varphi^x_c) \in \Delta$.

Then Δ^* is satisfiable in some structure $\mathfrak{A}$ such that $\overline{\overline{|\mathfrak{A}|}} \leq \overline{\overline{\mathbb{L}}}$.

Terminology: If $(\neg\dot{\forall} x\, \varphi \rightarrow \neg\varphi^x_c) \in \Delta$ we say that *c is a constant symbol for the pair* (φ, x).

Remark

If Δ satisfies 1-3 then:

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4. *If* $\alpha \in \Delta$ *and* $(\alpha \to \beta) \in \Delta$ *then* $\beta \in \Delta$;

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4. *If* $\alpha \in \Delta$ *and* $(\alpha \to \beta) \in \Delta$ *then* $\beta \in \Delta$;
5. $\alpha \in \Delta \Rightarrow \beta \in \Delta$ *iff* $(\alpha \to \beta) \in \Delta$;

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5. $\alpha \in \Delta \Rightarrow \beta \in \Delta$ *iff* $(\alpha \to \beta) \in \Delta$;
6. $(\alpha \to \beta) \in \Delta$ *iff* $(\neg\beta \to \neg\alpha) \in \Delta$;

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4. *If* $\alpha \in \Delta$ *and* $(\alpha \to \beta) \in \Delta$ *then* $\beta \in \Delta$;
5. $\alpha \in \Delta \Rightarrow \beta \in \Delta$ *iff* $(\alpha \to \beta) \in \Delta$;
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3. $\varphi \in \Delta$ *iff* $\Delta \vdash \varphi$;
4. *If* $\alpha \in \Delta$ *and* $(\alpha \to \beta) \in \Delta$ *then* $\beta \in \Delta$;
5. $\alpha \in \Delta \Rightarrow \beta \in \Delta$ *iff* $(\alpha \to \beta) \in \Delta$;
6. $(\alpha \to \beta) \in \Delta$ *iff* $(\neg\beta \to \neg\alpha) \in \Delta$;
7. *If* $\varphi_c^x \in \Delta$ *(where c is a constant symbol for the pair* (φ, x)*) then* $\dot{\forall} x\, \varphi \in \Delta$.

Towards the Proof of Lemma 1

Let Δ satisfy 1-3 of Lemma 1. Must find a structure $\mathfrak{A}$ and $s\colon V \longrightarrow |\mathfrak{A}|$ such that s satisfies Δ^* in $\mathfrak{A}$.

- $|\mathfrak{A}| =?$
- $f^{\mathfrak{A}} =?$ for f a function symbol;
- $R^{\mathfrak{A}} =?$ for f a relation symbol;
- $c^{|\mathfrak{A}} =?$ for c a constant symbol;
- $s =?$.

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- $|\mathfrak{A}| = \{t \mid t \text{ is a term of } \mathbb{L}\}$;
- $f^{\mathfrak{A}}(t_1, \ldots, t_n) = ft_1 \ldots t_n$ for f an n-ary function symbol (and terms $t_1, \ldots t_n$)
- $(t_1, \ldots, t_n) \in R^{\mathfrak{A}}$ iff $Rt_1 \ldots t_n \in \Delta$
  if R is an n-ary relation symbol other than $\dot{E}$;
- $(t_1, t_2) \in \dot{E}^{\mathfrak{A}}$ iff $t_1 \doteq t_2 \in \Delta$;
- $c^{\mathfrak{A}} = c$.

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Claim (2)

For every wff φ *of* $\mathbb{L}$,

$$\varphi \in \Delta \iff \models_{\mathfrak{A}} \varphi^*[s] .$$

Note that Claim 2 completes the proof of Lemma 1.