GÖDEL COMPLETENESS THEOREM

1. Properties of $\vdash$

Γ and Δ are sets of wffs of some first-order language L. The following statements follows directly from the definition of $\vdash$.

Proposition 1.1.

(1) *If* $\Gamma \vdash \varphi$ *then there is a finite subset* Δ *of* Γ *such that* $\Delta \vdash \varphi$*;*

(2) *If* $\alpha_1, \ldots, \alpha_n$ *is a deduction from* Γ *and* $\beta_1, \ldots, \beta_m$ *is a deduction from* γ*, then* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ *is also a deduction from* Γ.

(3) *If* $\varphi \in \Gamma \cup \Lambda$ *then* $\Gamma \vdash \varphi$.

(4) *If* $\Gamma \vdash \varphi$ *and* $\Gamma \subseteq \Delta$ *then* $\Delta \vdash \varphi$.

(5) *If* $\Gamma \vdash \alpha$ *and* $\Gamma \vdash \alpha \to \beta$ *then* $\Gamma \vdash \beta$.

(6) *If* $\Gamma \vdash \alpha$ *and* $\alpha \vdash \beta$ *then* $\Gamma \vdash \beta$.

(7) *If* $\Gamma \vdash \alpha$ *then for all* β,

$$\Gamma \vdash \beta \to \alpha \, .$$

(8) *If* $\Gamma \vdash \alpha \wedge \beta$ *then* $\Gamma \vdash \alpha$ *and* $\Gamma \vdash \beta$. *(Caution:* $\wedge$ *and "and" cannot be replaced by* $\vee$ *and "or" in the above statement.)*

(9) *If* $\Gamma \vdash \alpha$ *and* $\Gamma \vdash \beta$ *then* $\Gamma \vdash \alpha \wedge \beta$.

(10) *If* $\Gamma \vdash \alpha$ *or* $\Gamma \vdash \beta$ *then* $\Gamma \vdash \alpha \vee \beta$.

(11) *If* $\Gamma \vdash \alpha \to \beta$ *then* $\Gamma \cup \{\alpha\} \vdash \beta$.

The converse of (10) is also true. In fact:

Theorem 1.2. *(Deduction Theorem) If* $\Gamma \cup \{\alpha\} \vdash \beta$ *then* $\Gamma \vdash \alpha \to \beta$.

Furthermore, if $\beta_1, \ldots, \beta_n$ *is a deduction of* β *from* $\Gamma \cup \{\alpha\}$ *then there is a deduction* $\gamma_1, \ldots, \gamma_m$ *of* $\alpha \to \beta$ *from* Γ *such that no function symbol, relation symbol, or constant symbol appears in* $\gamma_1, \ldots \gamma_m$ *unless it also appears in the deduction* $\beta_1, \ldots, \beta_n$.

Theorem 1.3. *(Generalization Theorem) If* $\Gamma \vdash \varphi$ *and* x *does not appear free in (any member of)* Γ *then* $\Gamma \vdash \forall x\, \varphi$.

Theorem 1.4. *(Generalization on constants) Suppose* $\Gamma \vdash \varphi$ *and* c *is a constant symbol not in* Γ. *Then there is a variable* y *not in* φ *such that*

(1) $\Gamma \vdash \dot\forall y\, \varphi^c_y$, *and furthermore,*

(2) *if* $\alpha_1, \ldots, \alpha_n$ *is a deduction of* φ *from* Γ *then there is a deduction* $\beta_1, \ldots, \beta_m$ *of* $\dot\forall y\, \varphi^c_y$ *in which* c *does not appear, and such that no relation symbol, function symbol, or constant symbol appears in the deduction* $b_1, \ldots, \beta_m$ *unless it also appears in the deduction* $\alpha_1, \ldots, \alpha_n$.

Using this, we get

Theorem 1.5. *Let Γ be a consistent set of wffs of a language L, and let L' be obtained from L by adding to L a new set C of constant symbols. Then Γ is also a consistent set of wffs of L'.*

Theorem 1.6. *(Corollary 24G) Suppose $\Gamma \vdash \varphi^x_c$, where c is a constant symbol that does not appear in Γ or φ. Then $\Gamma \vdash \dot{\forall} x\, \varphi$ and there is a deduction of $\dot{\forall} x\, \varphi$ from Γ in which c does not appear*

Theorem 1.7. *(Alphabetic variants) Given a wff φ, term t, and variable x, there is a wff φ' that differs from φ only in quantified variables, such that:*

(1) *$\vdash \varphi \leftrightarrow \varphi'$, and*
(2) *t is substitutable for x in φ'.*

Proposition 1.8. *(Equality)*

(1) $\vdash x \doteq y \to y \doteq x$;
(2) $\vdash x \doteq y \to y \doteq z \longrightarrow x \doteq z$;
(3) *for every n-ary relation symbol P and variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$,*

$$\vdash x_1 \doteq y_1 \to \cdots \longrightarrow x_n \doteq y_n \to Px_1 \cdots x_n \to Py_1 \cdots y_n \;;$$

(4) *for every n-ary relation function symbol f and variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$,*

$$\vdash x_1 \doteq y_1 \to \cdots \to x_n \doteq y_n \to fx_1 \cdots x_n \doteq fy_1 \cdots y_n \;;$$

2. Gödel Completeness Theorem

An inspection of the set Λ of logical axioms shows that every member of Λ is valid. Note that if for formulas α and β, if $\models \alpha$ and $\models \alpha \to \beta$ then $\models \beta$; i.e. the rule of modus ponens preserves validity. An easy proof by induction then gives:

Theorem 2.1. *(Soundness) If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.*

Theorem 2.2. *(Gödel Completeness Theorem) If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.*

We actually prove the following generalization of the Gödel Completeness Theorem that gives information about the size of structures satisfying a set Γ.

Theorem 2.3. *(Extended Gödel Completeness Theorem) If Γ is a consistent set of wffs of the first-order language L then Γ is satisfied in a structure whose cardinality is not bigger than that of L.*

In particular, it follows that if Γ is a consistent set of wffs of an enumerable language then Γ is satisfied in some countable structure.

Proof. (Outline) Let L be a first-order language. If $\doteq$ is in L, let L' be the language obtained from L by removing $\doteq$ and adding a new binary relation symbol E. (If $\doteq$ is not in L, then L' is the same as L.) For each wff φ of L

let φ^* be the wff of L' obtained from φ by replacing every occurrence of $\doteq$ by E. For Δ a set of wffs of L, let

$$\Delta^* = \{\varphi^* : \varphi \in \Delta\} .$$

The following lemmas give an outline of the proof.

Lemma 2.4. *Let Δ be a set of wffs of L such that*

(1) *Δ is consistent;*
(2) *for every φ of L,*
$$\varphi \in \Delta \text{ or } (\neg\varphi) \in \Delta ;$$
(3) *for every wff φ of L and variable x, there is a constant symbol c such that*
$$(\neg\dot{\forall} x\, \varphi \to \neg\varphi^x_c) \in \Delta .$$

Then Δ^* is satisfiable in some structure whose cardinality is not bigger than that of L.

Lemma 2.5. *Let Δ be a set of wffs of L satisfying (1)-(3) of Lemma 2.4. Then Δ is satisfiable in some structure with cardinality less than that of L.*

Lemma 2.6. *Let Γ be a consistent set of wffs of L. There is a language L', obtained from L by adding a new set of constant symbols, such that the cardinality of L' is the same as that of L, and a set $\Delta \supseteq \Gamma$ of wffs of L' such that*

Δ is consistent, and
Δ satisfies (1)-(3) of Lemma 2.4.

We can now complete the proof of the Gödel Completeness Theorem. Let Γ be a consistent set of wffs of the first-order language L. By Lemma 2.6 there is an extension L' of L obtained by adding a new set of constant symbols, and a set $\Delta \supseteq \Gamma$ of wffs such that Δ is a consistent set of wffs of L', and satisfies properties (1)-(3) of Lemma 2.4. By Lemma 2.5 (for the language L'), Δ is satisfied in some structure $\mathfrak{A}$ of L' such that the cardinality of $\mathfrak{A}$ is not greater than that of L. Since $\Gamma \subseteq \Delta$, the structure $\mathfrak{A}$ also satisfies Γ. Let $\mathfrak{A}'$ be the reduct of $\mathfrak{A}$ obtained by "cutting back" to the language L. Since Γ is a set of wffs of the original language L, Γ is satisfied in the structure $\mathfrak{A}'$. □

The Gödel Completeness Theorem is an easy consequence of the Extended Gödel Completeness Theorem.

3. Consequences of the Extended Gödel Completeness Theorem

Remark 3.1. Combining the Soundness Theorem 2.1 and the Completeness Theorem 2.2 we have

$$\Gamma \models \varphi \iff \Gamma \vdash \varphi .$$

Then using Proposition 2.1(1),

Corollary 3.2. If $\Gamma \models \varphi$ then there is a finite subset Δ of Γ such that $\Delta \models \varphi$.

Theorem 3.3. *(Compactness Theorem) If every finite subset of Γ is satisfiable then Γ is satisfiable .*

Corollary 3.4. If $\Gamma \models \varphi$ then there is a finite subset Δ of Γ such that $\Delta \models \varphi$.

Theorem 3.5. *(Downwards Löwenheim-Skolem Theorem) Let Γ be a set of wffs of L. If Γ is satisfiable then Γ is sastisfiable in a structure $\mathfrak{A}$ with $\overline{\overline{|\mathfrak{A}|}} \leq \overline{\overline{L}}$.*

In particular, iff L is enumerable then if Γ is satisfiable, Γ is satisfiable in a countable structure.

Theorem 3.6. *(Upwards Löwenheim-Skolem Theorem) Let Γ be a set of wffs of L. If Γ is satisfiable in some infinite structure then for every cardinal $\kappa \geq \overline{\overline{L}}$, Γ is satisfiable in a structure of cardinality κ.*

Last semester you were asked to try to find a set Σ of sentences of some language L such that for each n, Σ has a model with at least n members but has no infinite model. Now you can see why you were unsucessful.

Proposition 3.7. *Let Σ be a set of sentences of L. If for each n, Σ has a model with at least n members then Σ has an infinite model.*

You should review from last semester definitions of :

$\mathfrak{A} \equiv \mathfrak{B}$ ($\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent*);
$\mathfrak{A} \cong \mathfrak{B}$ ($\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic*);
$\mathrm{Th}(\mathfrak{A})$ (the *theory* of $\mathfrak{A}$).

Last semester we showed that if $\mathfrak{A} \equiv \mathfrak{B}$ and $|\mathfrak{A}|$ is finite, then $\mathfrak{A} \cong \mathfrak{B}$. We can now see that we can not eliminate "finite" from this statement.

Proposition 3.8. $\mathfrak{A} \equiv \mathfrak{B}$ *iff* $\mathfrak{B}$ *is a model of* $\mathrm{Th}(\mathfrak{A})$.

Using this we get:

Proposition 3.9. *If $|\mathfrak{A}|$ is infinite then there is a structure $\mathfrak{B}$ such that $\mathfrak{A} \equiv \mathfrak{B}$ but $\mathfrak{A} \not\cong \mathfrak{B}$.*

4. Nonstandard models of the integers

From now on, σ is always a sentence, and Σ is always a set of sentences.

We now look at the structure $\mathfrak{N} = (\mathbb{N}, S, <, +, \times, 0)$. Here S is the successor function. The language has a one place function symbol $\dot{S}$ (denoting S), a binary relation symbol $\dot{<}$ (denoting $<$), a binary function symbol $\dot{+}$ (denoting $+$), a binary function symbol $\dot{\times}$ (denoting $\times$), and a constant symbol $\dot{0}$ (denoting 0).

Suppose we want to put down axioms for $\mathfrak{N}$ that describes $\mathfrak{N}$ as much as possible. What axioms do we want? A naive goal would be to get a set Σ of sentences such that the models of Σ are exactly the structures that are isomorphic to $\mathfrak{N}$. More precisely, we would like find a set Σ of sentences such that:

(1) $\mathfrak{N}$ is a model of Σ, and
(2) for every structure $\mathfrak{B}$, if $\mathfrak{B}$ is a model of Σ then $\mathfrak{N} \cong \mathfrak{B}$.

However, by the ULST there is no such set Σ. (For such a Σ would have an infinite model, $\mathfrak{N}$, and so by the ULST, Σ would then also have uncountable models.) Instead, we could try to find a set Σ of sentences such that all *enumerable* models of Σ are isomorphic. But we can't even do this.

Definition 4.1. A structure $\mathfrak{A}$ such that $\mathfrak{A} \equiv \mathfrak{N}$ but $\mathfrak{A} \ncong \mathfrak{N}$ is called a *nonstandard model of the integers*, or more precisely a *nonstandard model of* $\mathrm{Th}(\mathfrak{N})$.

Proposition 4.2. *There are enumerable nonstandard models of the integers.*

5. Theories

In this section we see how to answer the following question.

Question 5.1. Are the structures $\mathfrak{Q} = (\mathbb{Q}, <)$ and $\mathfrak{R} = (\mathbb{R}, <)$ elementarily equivalent?

Definition 5.2. A set of sentences Σ is a *theory* if for every sentence σ,

$$\Sigma \vdash \sigma \to \sigma \in \Sigma \ .$$

We often use T to stand for a theory.

Definition 5.3. For a set Σ of sentences,

$$\mathrm{Cn}(\Sigma) = \{\sigma : \Sigma \vdash \sigma\} \ .$$

Proposition 5.4. *For every set Σ of sentences, $\mathrm{Cn}(\Sigma)$ is a theory.*

Examples 5.5. (1) The set $\{\sigma :\models \sigma\}$ of valid sentences is a theory (since $\{\sigma :\models \sigma\} = \mathrm{Cn}(\emptyset)$).
(2) For every structure $\mathfrak{A}$, $\mathrm{Th}(\mathfrak{A})$ is a theory.

Definition 5.6. The theory T is *complete* if for every sentence σ

$$\sigma \in T \text{ or } (\neg\sigma) \in T \ .$$

Examples 5.7. (1) $\mathrm{Th}(\mathfrak{A})$ is a complete theory for every structure $\mathfrak{A}$.
(2) The set of valid wffs is a theory but not a complete theory.

Proposition 5.8. *The theory T is complete iff all models of T are elementarily equivalent.*

Definition 5.9. Let Σ consist of the following finite set of sentences of the language L with $\dot{<}$ and $\doteq$ that express:

(1) linear ordering (transitivity and trichotomy);
(2) the density property;
(3) there is no largest element, and there is no smallest element.

The theory $\mathrm{Cn}(\Sigma)$ is called *the theory of dense linear orderings without endpoints*, and a model of Σ is called a dense linear ordering without endpoints.

Exercise 5.10. (*Enderton, Exercise 4, p. 163*) Show that all enumerable dense linear orderings without endpoints are isomorphic.

Using DLST 3.5, Proposition 5.8, Exercise 5.10, and Proposition 3.8, we get:

Proposition 5.11. *The theory of dense linear orderings without endpoints is a complete theory.*

We can now answer Question 5.1 (1). Since $\mathfrak{Q}$ and $\mathfrak{R}$ are both dense linear orderings without endpoints:

Corollary 5.12. The structures $\mathfrak{Q}$ and $\mathfrak{R}$ are elementarily equivalent.

6. Reasonable Languages

Is there an algorithm, which given as input a sentence σ will give output *yes* if σ is true in $\mathfrak{Q}$ and output *no* if σ is false in $\mathfrak{Q}$?

More generally, given a structure $\mathfrak{A}$ is there such an algorithm? Does the answer depend on the structure? This question is particularly important for the structure $\mathfrak{N} = (\mathbb{N}, <, S, +, \times, 0)$ since there are many important mathematical statements that can be expressed in the language for this structure.

This type of question does not make sense if , for example, there is no algorithm for determining if a finite sequence of symbols of the language is a wff. To answer these questions we restrict ourselves to *reasonable languages*.

Definition 6.1. Let $p_0, \ldots, p_n, \ldots$ be the increasing enumeration of the prime numbers. We associate a number $\langle a_0, \ldots, a_n \rangle$ with each finite sequence $a_0, \ldots, a_n$ of members of $\mathbb{N}$ by letting

$$\langle a_0, \ldots, a_n \rangle = p_0^{a_0+1} \cdot p_1^{a_1+1} \cdot \cdots \cdot p_n^{a_n+1} .$$

We further define $\langle \emptyset \rangle = 1$.

Remark 6.2. This function $\langle \ \rangle$ is a one-to-one correspondence between all finite sequences and a subset of $\mathbb{N}$. It has the property that from a number $\langle a_0, \ldots, a_n \rangle$ we can effectively recover the finite sequence $a_0, \ldots, a_n$, and conversely, from a finite sequence $a_0, \ldots, a_n$ we can effectively finedthe number $\langle a_0, \ldots, a_n \rangle$.

Definition 6.3. (Informal) Let L be a first order language with enumerably many symbols. The language L is *reasonable* if there is a one-to-one function h whose domain is the set of all symbols of L, and whose range is a subset of $\mathbb{N}$ and which satisfies the following conditions:

$h(\dot{\forall}) = 0$
$h(\dot{0}) = 2$
$h(\dot{S}) = 4$
$h(\dot{<}) = 6$
$h(\dot{+}) = 8$
$h(\dot{\times}) = 10$
$h(\dot{E}) = 12$
$h(() = 1$
$h()) = 3$
$h(\neg) = 5$
$h(\rightarrow) = 7$
$h(\doteq) = 9$
$h(v_1) = 11$
$h(v_2) = 13$

We further require:

(1) The set $\{(h(P), n) : P \text{ is an } n\text{-ary relation symbol}\}$ is effectively decidable;
(2) The set $\{(h(f), n) : f \text{ is an } n\text{-ary function symbol}\}$ is effectively decidable;
(3) The set $\{h(c) : c \text{ is a constant symbol}\}$ is effectively decidable.

Remark 6.4. (1) If some of the above symbols (such as $\dot{0}$ for example) are not symbols of L then h is not defined on these symbols.
(2) All of the odd numbers are used up. So the value of h at a function symbol, relation symbol, or constant symbol must be even.
(3) If the set of relation symbols, function symbols, and constant symbols is finite then L is reasonable. In particular the language for the structure $\mathfrak{R}$ is reasonable, and the language for the structure $\mathfrak{N}$ is reasonable, since these languages have only finitely many relation symbols, function symbols, and constant symbols.

Definition 6.5. (1) Let $\epsilon = s_0 \cdots s_n$ be an expression of L (i.e. a finite sequence of symbols of L. We associate an integer $\#(\epsilon)$ with ϵ by letting

$$\#(\epsilon) = \#(s_0 \cdots s_n) = \langle h(s_0), \ldots, h(s_n) \rangle \ .$$

$\#(\epsilon)$ is called the Gödel number of ϵ.
(2) We associate with a finite sequence $\epsilon_0, \ldots \epsilon_n$ of expressions the integer

$$\mathcal{G}(\epsilon_0, \ldots, \epsilon_n) = \langle \#(\epsilon_0), \ldots, \#(\epsilon_n) \rangle \ .$$

$\mathcal{G}(\epsilon_0, \ldots, \epsilon_n) = \langle \#(\epsilon_0), \ldots, \#(\epsilon_n) \rangle$ is called the Gödel number of the finite sequence of expressions.

Let L be a reasonable language. A term of L is a finite sequence of symbols. Similarly, a wff of L is a finite sequence of symbols. Thus each term and each wff has a Gödel number.

Definition 6.6. (Informal) A set of terms, or a set of wffs is *effectively enumerable* (respectively, *effectively decidable*) if its set of Gödel numbers of members is an effectively enumerable (respectively, effectively decidable) set of integers.

Proposition 6.7. *Let L be a reasonable language.*

(1) *The set of terms of L is effectively decidable.*
(2) *The set of wffs of L is effectively decidable.*
(3) *The set of sentences of L is effectively decidable.*
(4) *The set Λ of logical axioms is effectively decidable.*

A finite sequences of wffs is a finite sequence of expressions and so has a Gödel number.

Definition 6.8. (Informal) Let L be a reasonable language. A finite sequence of wffs is *effectively enumerable* (respectively, *effectively decidable* if its set of Gödel numbers is effectively enumerable (respectively, effectively decidable).

Informally, a set Γ of wffs is effectively decidable if there is an algorithm for determining membership in Γ, and Γ is effectively enumerable if there is an algorithm for listing the members of Γ.

Similarly, a set $\mathcal{X}$ of finite sequences of wffs is effectively decidable if there is an algorithm, which on input a finite sequence of wffs will give output *yes* if the finite sequence belows to $\mathcal{X}$ and output *no* if the finite sequence does not below to $\mathcal{X}$..

7. Axiomatizable Theories

We assume that the first-order language L is reasonable.

Proposition 7.1. *Let Γ be an effectively decidable set of wffs. Then the set of all deductions from Γ is effectively decidable.*

Corollary 7.2. If Γ is an effectively decidable set of wffs then the set of logical consequences of Γ is effectively enumerable.

This has the surprising consequence:

Corollary 7.3. The set of valid wffs is effectively enumerable.

Question 7.4. Is the set of valid wffs effectively decidable? Does the answer depend on the reasonable language L?

Definition 7.5. (Informal) Let T be a theory.

(1) T is *axiomatizable* if there is an effectively decidable set of sentences Σ such that $\mathrm{Cn}(\Sigma) = T$.
(2) T is *finitely axiomatizable* if there is a finite set of sentences Σ such that $\mathrm{Cn}(\Sigma) = T$.

Since every finite set of sentences is effectively decidable, every finitely axiomatizable theory is also axiomatizable.

Theorem 7.6. *If the theory T is axiomatizable then T is effectively enumerable.*

Complete theories are of special interest for the following reason.

Theorem 7.7. *If the theory T is axiomatizable and complete then T is effectively decidable.*

Since the theory of dense linear orderings without endpoints is a finitely axiomatizable theory, and we have shown that it is complete:

Corollary 7.8. The theory of dense linear orderings without endpoints is effectively decidable.

Since the theory of dense linear orderings without endpoints is the same as $\mathrm{Th}(\mathfrak{R})$ and $\mathrm{Th}(\mathfrak{Q}$, these theories are effectively decidable.

8. Elimination of Quantifiers

We next look at some simple structures associated with elementary number theory, and ask the same questions about decidability and definability.

Definition 8.1. Let:

$$
\begin{aligned}
\mathfrak{N}_S &= (\mathbb{N}, S, 0)\ ; \\
\mathfrak{N}_L &= (\mathbb{N}, <, S, 0)\ ; \\
\mathfrak{N}_+ &= (\mathbb{N}, < S, +, 0)\ ; \\
\mathfrak{N}_{+,\times} &= (\mathbb{N}, < S, +, \times, 0)\ ; \\
\mathfrak{N}_{+,\times,E} &= (\mathbb{N}, <, S, +, \times, E, 0)\ ,
\end{aligned}
$$

where S is the successor function on $\mathbb{N}$, and

$$E(m, n) = m^n$$

is the exponential function on $\mathbb{N}$ (where $0^0 = 1$).

Question 8.2. For each of the above structures:

(1) Is the associated theory effectively decidable?
(2) What subsets of the universe are definable in the structure?

Note that if the theory in question is axiomatizable then by Theorem 7.7 (since the theory of a structure is complete) the theory is effectively decidable. We start by putting down axioms for the structure $\mathcal{N}_S$.

Definition 8.3. (Axioms A_S)) l The axioms A_S are the following infinite list:

$$S1: \quad \forall x(\dot{S}x \neq \dot{0})$$
$$S2: \quad \forall x \forall y\,(\dot{S}x \dot{=} \dot{S}y \rightarrow x \dot{=} y)$$
$$S3: \quad \forall x\,(x \neq \dot{0} \rightarrow \exists y\,(\dot{S}y \dot{=} x))$$
$$S4.1 \quad \forall x\,(\dot{S}x \neq x)$$
$$S4.2 \quad \forall x\,(\dot{S}\dot{S}x \neq x)$$
$$\vdots$$
$$S4.n \quad \forall x\,\dot{S}^n x \neq x)$$
$$\vdots$$

Since each of these sentences is true in the structure $\mathfrak{N}_L$, we have $\mathrm{Cn}(A_S) \subseteq \mathrm{Th}(\mathfrak{N})_S$.

Definition 8.4. The theory T *admits elimination of quanitifiers* if for every wff φ there is a quantifier-free wff ψ such that $T \vdash \varphi \leftrightarrow \psi$.

Proposition 8.5. *Let T be a theory. Suppose that for every wff φ of the form $\exists x\,(\alpha_0 \wedge \cdots \wedge \alpha_n)$, where each α_i is either atomic or the negation of an atomic wff, there is a quantifier-free ψ such that $T \vdash \varphi \leftrightarrow \psi$. Then T admits elimination of quantifiers.*

Theorem 8.6. $\mathrm{Cn}(A_S)$ *admits elimination of quantifiers. In fact there is an algorithm that for every wff φ finds a quantifier-free wff ψ such that:*

(1) $T \vdash \varphi \leftrightarrow \psi$, *and*
(2) *every variable that occurs free in ψ also occurs free in φ. (In particular, if ϕ is a sentence, then so is ψ.)*

Exercise 8.7. Show that for every quantifier-free sentence σ,

$$A_S \vdash \sigma \ \text{ or } A_s \vdash \neg\sigma$$

Using Exercise 8.7, we have:

Corollary 8.8. (1) The theory $\mathrm{Cn}(A_S)$ is complete and effectively decidable.
(2) A subset of $\mathbb{N}$ is definable in $\mathfrak{N}_S$ iff it is finite or cofinite.

Is $\mathrm{Cn}(A_S)$ finitely axiomatizable?

Proposition 8.9. *For every theory* $\mathrm{Cn}(\Sigma)$ *(of any language), if* $\mathrm{Cn}(\Sigma)$ *is finitely axiomatizable then there is a finite subset Σ' of Σ such that* $\mathrm{Cn}(\Sigma') = \mathrm{Cn}(\Sigma)$.

Exercise 8.10. Use Proposition 8.9 to show that $\mathrm{Cn}(A_S)$ is not finitely axiomatizable.

Exercise 8.11. (1) Show that the theory of dense linear orderings without endpoints admits elimination of quantifiers. If you prefer, you can show that for every wff φ there is a quantifier-free wff ψ such that for every $s\colon V \longrightarrow \mathbb{N}$,

$$\models_{(\mathbb{R},<)} (\varphi \leftrightarrow \psi)[s] \ .$$

(2) Conclude that $\emptyset$ and $\mathbb{R}$ are the only definable subsets of $(\mathbb{R}, <)$.

Theorem 8.12. $\mathrm{Th}(\mathfrak{N}_L)$ *admits elimination of quantifiers. It follows that a subset of* $\mathbb{N}$ *is definable in* $\mathfrak{N}_L$ *iff it is finite or cofinite.*

Proof. See Enderton □

Remark 8.13. There is a wff φ that defines in $\mathfrak{N}_+$ the set of even nonnegative integers, which is neither finite nor cofinite. It follows that$\mathrm{Th}(\mathfrak{N}_+)$ does not admit elimination of quantifiers.

In spite of Remark 8.13:

Theorem 8.14. $\mathrm{Th}(\mathfrak{N}_+)$ *is effectively decidable.*

Proof. See Enderton □

You have already done the following exercise

Exercise 8.15. $+$ is not definable in the structure $\mathfrak{N}_\times$.

Rather difficult is the following:

Theorem 8.16. $\mathrm{Th}(\mathfrak{N}_\times)$ *is effectively decidable.*

This leads to:

Question 8.17. Is $\mathrm{Th}(\mathfrak{N}_{+,\times})$ effectively decidable?

This is an important question since many important unsolved mathematical statements about the nonnegative integers can be expressed as sentences in the language for this structure.