5165: Notes on Sentential Logic

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1 Sentential Logic

1.1 Well-formed Formulas and Mathematical Induction

Definition 1.1. A *well-formed sequence* is a finite sequence $\alpha_1, \ldots, \alpha_n$ of expressions such that each α_i is either:

1. a sentence symbol
2. $(\neg\alpha_j)$, for some $j < i$, or
3. $(\alpha_j \to \alpha_k)$, for some $j, k < i$, or
4. $(\alpha_j \vee \alpha_k)$, for some $j, k < i$, or
5. $(\alpha_j \wedge \alpha_k)$, for some $j, k < i$, or
6. $(\alpha_j \leftrightarrow \alpha_k)$, for some $j, k < i$.

An expression α is a *well-formed formula* (wff) if there is a well-formed sequence $\alpha_1, \ldots, \alpha_n$ such that $\alpha = \alpha_n$.

Remark 1.2. It is clear that every non-empty initial subsequence of a well-formed sequence is also a well-formed sequence. It follows that if $\alpha_1, \ldots, a_n$ is a well-formed sequence, then each expression α_i in the sequence is a well-formed formula.

Proposition 1.3. *Every wff has one of the following forms:*

$$A,\ (\neg\alpha),\ (\alpha\wedge\beta),\quad (\alpha\vee\beta),\quad (\alpha\to\beta),\quad (\alpha\leftrightarrow\beta)\ , \tag{1}$$

where A is a sentence symbol and α and β are wffs.

Proof. Suppose α is a wff. By definition, there is a well-formed sequence

$$\alpha_1, \ldots, \alpha_n$$

such that $\alpha_n = \alpha$. By the definition of a well-formed sequence, each α_i, for $i \leq n$ is of one of the forms 1-5 of Definition 1.1. In particular, α_n has one of those forms. By Remark 1.2, each α_i for $i \leq n$ is a wff and so α_n has one of those forms, where α_i and α_j is a wff. □

Proposition 1.4. *Let S be a set of wffs. If*

1. *Every sentence symbol is in S, and*
2. *For every wff α and wff β,*

   if α and β are in S then each of $(\neg\alpha)$, $(\alpha\to\beta)$, $(\alpha\vee\beta)$, $\alpha\wedge\beta)$, and $(\alpha\leftrightarrow\beta)$ is in S,

   then S is the set of all wffs.

Proof. Suppose S is a set of wffs which satisfies (a) and (b). Let α be a wff. We show that α is in S. Since α is a wff, there is a well-formed sequence $\alpha_1, \ldots, a_n$ such that $\alpha = \alpha_n$. We show by induction on i, for $1 \leq i \leq n$, that $\alpha_i \in S$.

1. *Basis:* $i = 1$. Then α is a sentence symbol, and so $\alpha \in S$ by 1 above. *Induction Step:* $i > 1$. By the induction hypothesis, $\alpha_j \in S$ for every i such that $1 \leq j < i$. Since $\alpha_1, \ldots, \alpha_n$ is a well-formed sequence, it satisfies properties 1-6 above.

2. *Case (i):* α_i is a sentence symbol. Then $\alpha_i \in S$ by 1.

3. *Case (ii):* α_i is $(\neg\alpha_j)$, for some $j < i$. By induction hypothesis, $\alpha_j \in S$. But then by 2 above, $\alpha_i = (\neg\alpha_j) \in S$.

4. *Case (iii):* α is $(\alpha_j \to \alpha_k)$, for some $j, k < i$. By the induction hypothesis, both α_j and α_k are in S. But then by 2, $\alpha_i = (\alpha_j \to \alpha_k) \in S$.

The other cases are similar. $\square$

Exercise 1.5 (Optional). Let $\mathcal{P}$ be the following collection of sets S of expressions:

$S \in \mathcal{P} \iff S$ is a set of expressions & S satisfies (a) and (b) of Prop. 2

Show that $\bigcap \mathcal{P}$ is the set of all wffs.[1]

1.2 Logically Valid Sentences and Logically Correct Reasoning

The sentence

The current outdoor temperature in Minneapolis is less than 99 degrees.

happens to be a true sentence. To realize that it is true one must understand the meaning of the words (temperature, etc.) in the sentence (as well as know something about the current temperature).

The sentence

The current outdoor temperature in Minnesota is less than 99 degrees, or it is not the case that the current outdoor temperature is less than 99 degrees.

also is a true sentence. However, in this case the truth of the sentence does not depend upon understanding the meaning of the words (temperature, etc.). Rather, the sentence is seen to be true simply by looking at the *form* of the sentence. This sentence has the form $A \vee \neg A$ and *every* such sentence is true. Such a sentence is called *logically valid*, or simply *valid*. Next consider

[1] *Note:* $\bigcap \mathcal{P}$ is the intersection of all members of $\mathcal{P}$. So for example, if $\mathcal{P} = \{A, B\}$, then

$$\bigcap \mathcal{P} = \bigcap \{A, B\} = A \cap B \ ,$$

and if $\mathcal{P} = \{A_1, \ldots, A_n, \ldots\}$, then

$$\bigcap \mathcal{P} = \bigcap \{A_1, \ldots, A_n, \ldots\} = \bigcap_{n=1}^{\infty} A_n \ .$$

the following form of reasoning:

Premise: The current outdoor temperature in Minneapolis is below 32 degrees.

Conclusion: It is snowing.
Now the conclusion might be true, or it might be false. Next consider the reasoning:

Premise: The current outdoor temperature in Minneapolis is below 32 degrees.

Conclusion: The current outdoor temperature in Minneapolis is below 32 degrees, or it is snowing.

The conclusion might be true, or it might be false. However, in this case the reasoning is seen to be correct, since *if* the premise is true, then the conclusion *must* also be true. We say in this case that the reasoning is *logically valid*, or simply *valid*. The fact that reasoning is logically valid does not mean that the conclusion of the reasoning is true. It simply means that *if* the premises (i.e. hypotheses) are true, then the conclusion must also be true. One of our goals is to gain insight into the *form* of reasoning which is valid.

Exercise 1.6. Determine whether the conclusion is correctly inferred from the premises, by making a suitable translation into the language of Sentential Logic.

Premises:

(a) If Jones did not meet Smith last night, then either Smith is the murderer, or Jones is lying.

(b) If Smith is not the murderer, then Jones did not meet Smith last night, and the murder took place after midnight.

(c) If the murder took place after midnight, then either Smith is the murderer or Jones is lying.

Conclusion:

Smith is the murderer.

Premises:

(a) If capital investment remains constant, then government spending will increase or the level of unemployment will increase.

(b) If government spending does not increase, then taxes will be reduced.

(c) If taxes are reduced and capital investment remains constant, then the level of unemployment will not increase.

Conclusion:

Government spending will increase.

1.3 Disjunctive Normal Form and Conjunctive Normal Form

Exercise 1.7. Find a wff α in disjunctive normal form, and a wff β in conjunctive normal form which are equivalent to the given wff.

1. $\neg A \longrightarrow (\neg B \vee C) \longrightarrow A$
2. $((\neg A \longrightarrow A \wedge B) \longrightarrow C) \longrightarrow B \wedge A$
3. $\neg(A \longrightarrow B) \longrightarrow C$

Definition 1.8. The fact that every wff is tautologically equivalent to a wff in disjunctive normal form suggests an alternative way of defining the language of sentential logic. Instead of using all the connectives $\longrightarrow$, $\longleftrightarrow$, $\neg$, $\vee$, and $\wedge$, we can get by with just $\neg$, $\vee$ and $\wedge$. In this exercise we assume that only these three sentential connectives are used to define the wffs.

1. Let v_1 and v_2 be truth assignments. $v_1 \preceq v_2$ iff for every sentence symbol A
$$v_1(A) = T \implies v_2(A) = T \; . \tag{2}$$
2. The wff φ is *monotone* if for every pair v_1 and v_2 of truth assignments,
$$\bar{v_1}(\varphi) = T \;\&\; v_1 \preceq v_2 \implies \bar{v_2}(\varphi) = T \; . \tag{3}$$

3. The wff φ is *positive* if there are no occurences of $\neg$ in φ, i.e. it has at most the connective symbols $\vee$ and $\wedge$.

Remark 1.9. *monotonicity* is a semantic property, i.e. it is defined in terms of truth values. *positivity* is a syntactic property, i.e. it is defined in terms of the form of the wff.

Exercise 1.10. Show that every positive wff is monotone.

Not every monotone wff φ is positive. For example, the wff $\neg\neg A$ is monotone but not positive. It is, however, tautologically equivalent to the wff A, which is positive.

Exercise 1.11. Is every monotone wff φ tautologically equivalent to a positive wff? If you think the answer is no, try to find a counterexample . If you think the answer is yes, try to prove it.

Exercise 1.12. Let α and β be wffs, such that:

(a) α has only the sentence symbols A and B,

(b) β has only the sentence symbols A and C, and

(c) $\alpha \models \beta$.

1. Describe a method for finding a wff γ that has just the sentence symbol A, and such that
$$\alpha \models \gamma \text{ and } \gamma \models \beta \ .$$
2. Use 1. to conjecture a more general statement, and try to prove it. :-)

1.4 Topology and the Compactness Theorem

Let
$$2^{\mathbb{N}} = \{f : f \colon \mathbb{N} \longrightarrow \{0,1\}\} \ .$$
Note that there is a one-to-one correspondence between $2^{\mathbb{N}}$ and the set of truth assignments. (For v a truth assignment, let
$$f_v(n) = ? \ .)$$
Let the set $\{0,1\}$ have the discrete topology, and the set $2^{\mathbb{N}}$ have the product topology. Note that the set $2^{\mathbb{N}}$ is compact. (why?) With each set Δ of wffs we can can associate a subset X_Δ of $2^{\mathbb{N}}$, where
$$X_\Delta = \{f_v : v \text{ satisfies } \Delta\} \ .$$

Exercise 1.13 (For Math majors). Use the fact that $2^{\mathbb{N}}$ is a compact space to give a proof of the Compactness Theorem.

1.5 Infinite Trees and the Compactness Theorem

We can identify with each finite sequence of 0's and 1's a unique node on the following infinite tree $\mathbb{T}$: An infinite path p through the tree $\mathbb{T}$ corresponds to a unique infinite sequence of 0's and 1's. This in turn defines a unique function $f \in \mathbb{N}$, where

$$\begin{aligned} &f(0) \text{ is the first term of } p \\ &f(1) \text{ is the second term of } p \\ &\quad\vdots \\ &f(n) \text{ is the } n+1\text{-st term of } p \\ &\quad\vdots \end{aligned}$$

Thus each infinite path p through the tree $\mathbb{T}$ is identified with a unique member f of $2^{\mathbb{N}}$. Conversely, a function $f \in 2^{\mathbb{N}}$ defines a unique infinite sequence of 0's and 1's, namely the infinite sequence $(f(0), f(1), \ldots, f(n), \ldots)$, and this in turn is associated with a unique path p through the $\mathbb{T}$. Thus there is a one-to-one correspondence between $2^{\mathbb{N}}$ and paths through the tree $\mathbb{T}$.

Now consider a subtree $\mathbb{S}$ of the tree $\mathbb{T}$ obtained by chopping off certain branches of $\mathbb{T}$. $\mathbb{S}$ is completely determined by specifying those finite sequences which are nodes of $\mathbb{S}$. In order for $\mathbb{S}$ to be a tree it must have the following property: If the finite sequence $(s_1, \ldots, s_n)$ is a node on $\mathbb{S}$, then every finite subsequence $(s_1, \ldots, s_i)$, where $1 \leq i < n$, is also a node on $\mathbb{S}$. Note that a subtree $\mathbb{S}$ of $\mathbb{T}$ might have only a finite number of nodes (in which case there are no infinite paths through $\mathbb{S}$).

Definition 1.14. Let $\mathbb{S}$ be a subtree of $\mathbb{T}$. A *path of length n on* $\mathbb{S}$ is a finite sequence $(s_1, \ldots, s_n)$ which is a node of $\mathbb{S}$.

Note that if $(s_1, \ldots, s_n)$ is a path of length n on $\mathbb{S}$, then $(s_1, \ldots, s_i)$ is a path of length i on $\mathbb{S}$ for $1 \leq i < n$.

Exercise 1.15 (König's Infinity Lemma). Let $\mathbb{S}$ be a subtree of the tree $\mathbb{T}$. Show that if for each $n \in \mathbb{N}$, $\mathbb{S}$ has a path of length n, then $\mathbb{S}$ has an infinite path.

Exercise 1.16. Use König's Infinity Lemma to prove the Compactness Theorem.

Exercise 1.17. Use the Compactness Theorem to prove König's Infinity Lemma.