# Math 5594H — Final Project

due Monday, December 15

You should complete a project dealing with differential equations and dynamical systems theory or its applications. This handout contains a few suggestions but its main purpose it to give you an idea of the possibilities. You should feel free to make up your own project. If you have an idea and want to check it out first, just stop by my office.

Projects can be divided into several basic types: papers about math, papers about applications and computer programs. I use the following criteria in coming up with grades. For math papers: correctness and depth (50), use of references (20), exposition (30). For papers about applications: use of references (30), analysis/critique (30), exposition (40). For computer projects: design and accuracy (50), use of program (30), exposition (20). For math projects, depth means going beyond the math of the course and getting into the details of your topic. For papers on applications, you need to look at several references; then analyze what you find, perhaps illustrating the points with your own examples. For computer projects, you should write the program for a specific purpose and then really use it for that purpose. It should be flexible enough to allow you to handle several examples.

**Mathematical Theory.** A project in this area might by based on one of the explorations at the end of a chapter of the text or you can investigate a related topic not covered in the text. Your paper could give an overview of the topic with some proofs that you have worked out. Or you could combine a theoretical exposition with some examples you have carried out using the computer.

1. Explorations. One notable feature of the text is the inclusion of "exploration" sections at the end of the chapters. These are presented as step by step investigations where you have to fill in all the details. Some of the more interesting and accessible topics include

- Ch. 8 Complex vectorfields, These are special cases of vectorfields in the plane arising from complex-differentiable functions. Additional material on complex vectorfields can be found in geometrically oriented books on complex variable theory such as
  Needham, "Visual complex analysis", Oxford University Press 1997; especially chapter 10.
- Ch. 10 Oscillating chemical reactions. Have a look at some of the original papers too.
- Ch. 11 Competing species with harvesting.
- Ch. 12 The Fitzhugh-Nagumo equations modelling the firing of nerve cells.
- Ch. 14 The Rössler attractor. A strange attractor similar to the Lorenz attractor.

2. Winding numbers and applications. A two-dimensional autonomous ODE is given by a vectorfield $F(x, y) = (f(x, y), g(x, y))$. Given such a vectorfield and a closed curve, $C$, in the plane, one can define a winding number of $F$ along $C$. The winding number can be used to prove existence of equilibrium points inside $C$. It can also used to prove the two-dimensional Brouwer fixed point theorem and the fundamental theorem of algebra. See, for example,

Chinn and Steenrod, "First Concepts of Topology", Math. Assoc. Amer. (1966), part II.

Henle, "A Combinatorial Introduction to Topology" , Dover (1994), chapter 2.

3. Series solutions and special functions. One important topic we will not cover in class is how to solve second order linear equations with nonconstant coefficients. These arise in physics and engineering when certain partial differential equations are solved using a version of the method of separation of variables. Many of these equations are named after some famous mathematician or physicist: Legendre's equation, Bessel's equation, etc. The corresponding solutions are called "special functions". Look into how such differential equations arise, how they can be solved using power series and how they sometimes give rise to "orthogonal polynomial" solutions.

Simmons, "Differential Equations with Applications and Historical Notes", McGraw-Hill (1972).

**Applications.** Write a survey paper describing an application of differential equations and dynamical systems. Explain how the ODE theory can be used to attack the problem. Try out some examples of your own. Start with the references below (or come up with your own topic) and and find other sources yourself, perhaps on the internet.

4. Synchronization of oscillators is an interesting phenomenon with many applications in biology (most of the cells in your heart contract at the same time).

Strogatz, "Nonlinear Dynamics and Chaos", Addison-Wesley (1994).

Strogatz, "Sync, the Emerging Science of Spontaneous Order", Hyperion (2003).

Glass, L. and Mackey, M.C., "From Clocks to Chaos: The Rhythms of Life", Princeton University Press (1988) .

5. Central forces and quasiperiodic orbits. The motion of a mass point in the plane under the influence of a central force is one of the classic problems of elementary mechanics. The most famous central force problem is the Kepler problem describing the motion of a planet around the sun, but this turns out to be a very atypical case in that all the solutions are periodic (the familiar elliptical motions). Other force laws give rise to quasiperiodic solutions. Look into this question. Explain the motion using two-dimensional tori. Investigate the frequency ratios on on these tori for various force laws. Why is the Kepler problem so special ?

Chapter 13 of the text.

Arnold, "Mathematical Methods of Classical Mechanics", Springer-Verlag (1989), chapter 2.

**Experiment.** Develop some programs for studying ODEs using either a high-level language like Mathematica or your favorite low-level language. Then use your program for something interesting.

6. Investigate the effectiveness of various numerical methods for solving ordinary differential equations. Compare, for example, the Euler method, the fourth-order Runge-Kutta method and the Adams-Moulton predictor-corrector method. Give some examples to show how fast they converge for different kinds of ordinary differential equations and for various stepsizes. On a more theoretical note, why do these methods converge at all ?

Press, et.al., "Numerical Recipes in C", Cambridge (1988), chapter 15.

Stoer, Bulirsch, "Introduction to Numerical Analysis", Springer (1993), chapter 7.

7. Write a program for computing and displaying Poincaré maps of periodically forced ODEs. For example, in a one-dimensional ODE, $x' = f(x,t)$ where $f(x, t+T) = f(x,t)$, the Poincaré map $P(x_0)$ is obtained by integrating the solution with initial condition $x_0$ for time $T$ to obtain the next point $x_1 = P(x_0)$. Then $x_2 = P(x_1)$ and so one. You should be able to compute these sequences and also to plot the graph of $x$ versus $P(x)$. Use your program to locate fixed points of $P$ (or equivalently, periodic orbits of the ODE) for some nontrivial examples. In two-dimensions, an ODE $(x', y') = (f(x,t), g(x,t))$ would lead to a map $(x_1, y_1) = P(x_0, y_0)$. Here you will probably have to be content with just plotting the sequence of points $(x_n, y_n)$. Try it out on some interesting examples like a periodically forced nonlinear pendulum or the periodically forced van der Pol equation.

8. Write a program which can display strange (or nonstrange) attractors for three dimensional flows. Write it so you can look from different directions and zoom in to show the fractal structure. Use it to study how the Lorenz attractor or the Rössler attractor evolves from simple to strange as the parameters are varied.