Math 5335 Section 2 Fall 2005
Solutions to December 1 homework problems

PROBLEM 9.3. By axiom I3 there exist points A, B, and C such that no line is incident with all three. Now, let $\mathcal{L}$ be our given line. Then $\mathcal{L}$ is incident with at most two of the three points A, B, and C. The remaining point or points is/are not incident with $\mathcal{L}$.

Note: What is presented above is a complete solution. But just to explain a bit more fully, suppose (for instance) that A and B are incident with $\mathcal{L}$. Then C is not incident with $\mathcal{L}$. Similarly, if A is the only one of the three that is incident with $\mathcal{L}$, then B and C are not incident with $\mathcal{L}$.

PROBLEM 9.9. Let's say that a "dual line" is an ordinary point in the 7-point geometry, and that a "dual point" is an ordinary line in the 7-point geometry. Then we have to show that the "dual points" and "dual lines" satisfy axioms I1, I2, and I3. This means that the ordinary points and lines have to satisfy the following statements:

J1. For any two lines $\mathcal{L}$ and $\mathcal{M}$, there is a unique point that is incident with both $\mathcal{L}$ and $\mathcal{M}$.
J2. Every point is incident with at least two lines.
J3. There exist three lines such that no point is incident with all three.

Now, let's prove that these hold for the 7-point geometry, ***starting with J3***. Since J3 just asserts that one or more instances exist, it ***suffices to present a particular instance***, such as the three lines that form the edges of the diagram in Figure 9.1 Thus, we would have
$\mathcal{L}_1 = \{(1,0,0), (1,1,0), (0,1,0)\}$, $\mathcal{L}_2 = \{(1,0,0), (1,0,1), (0,0,1)\}$, and
$\mathcal{L}_3 = \{0,1,0), (0,1,1), (0,0,1)\}$. Therefore $\mathcal{L}_1 \cap \mathcal{L}_2 = \{(1,0,0)\}$, $\mathcal{L}_1 \cap \mathcal{L}_3 = \{(0,1,0)\}$, and
$\mathcal{L}_2 \cap \mathcal{L}_3 = \{(0,0,1)\}$, so that $\mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3 = \varnothing$. The last equality of sets verifies our instance of J3.

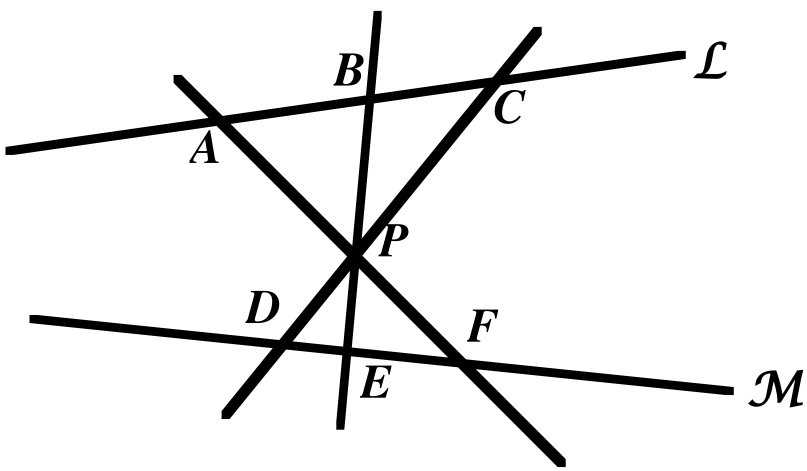
Here's one way to handle J2, but it's probably not the only way. Let P be the given point, and let $\mathcal{L}$ be a line not incident with P. This exists by Problem 9.3, which can be applied since the 7-point geometry is an incidence geometry. Now, if A and B are two distinct points of $\mathcal{L}$ (which exist by axiom I2), then $\overline{PA}$ and $\overline{PB}$ are distinct lines that are incident with P. {***Further explanation:*** We're using the notation $\overline{PA}$ to represent the unique line in the 7-point geometry that's incident with A and B, and similarly for $\overline{PB}$. Then $\overline{PA}$ and $\overline{PB}$ have to be distinct, since the equality $\overline{PA} = \overline{PB}$ would imply that $\overline{PA} = \overline{PB} = \overline{AB}$, thereby forcing P to be incident with $\mathcal{L} = \overline{AB}$, which is a contradiction.}

Now, finally we consider J1. Now, as we could have said in connection with J2, this is a finite situation, so that in principle we could handle it by just enumerating all of the cases. This would be rather tedious, since there are 7 lines and therefore $\binom{7}{2} = 21$ pairs of distinct lines. So, it is preferable to find a way to organize things somewhat. ***One approach*** involves assigning names such as $x, y,$ and z to the coordinates used in labeling the points. Then the edges discussed above are described as follows:

$$\mathcal{L}_1 \text{ is } z = 0, \qquad \mathcal{L}_2 \text{ is } y = 0, \qquad \text{and} \qquad \mathcal{L}_3 \text{ is } x = 0.$$

Each line that corresponds to a diagonal in the diagram is defined by an equation that sets two coordinates equal to each other, for instance $x = y,$ or $x = z,$ or $y = z.$ And finally, the "circular line" can be described by the equation $x + y + z = 2.$ {If you do addition modulo 2, then it becomes $x + y + z = 0.$} Anyway, you have to consider the cases of intersecting (1) two edges, or (2) two diagonals, or (3) an edge with a diagonal, or (4) intersecting the "circular line" with a line of another type. And you may end up having to break case (3) into two sub-cases.

A somewhat different approach involves ***proof by contradiction.*** Thus suppose that there are two lines, say $\mathcal{L}$ and $\mathcal{M}$, with no points in common. There are 3 points on each line in this geometry. Therefore, $\mathcal{L}$ and $\mathcal{M}$ jointly account for 6 of the 7 points of the 7 point geometry. Let P be the point that is incident with neither $\mathcal{L}$ nor $\mathcal{M}$. Let's assume that $\mathcal{L} = \{A,B,C\}$ and $\mathcal{M} = \{D,E,F\}$. [In other words, we're just assigning names to the points of $\mathcal{L}$ and $\mathcal{M}$.] Consider the lines $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$. Since each of these lines contains a third point, that point has to be a point of $\mathcal{M}$. So, the lines $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ coincide with the lines $\overline{PD}$, $\overline{PE}$, and $\overline{PF}$ in some order. {In the diagram below, a particular choice was made.}



Therefore the assumption that $\mathcal{L}$ and $\mathcal{M}$ have no points in common leads to the conclusion that the five lines $\mathcal{L}$, $\mathcal{M}$, $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ are the only lines in the 7-point geometry. But this contradicts the fact that there are seven lines in the 7-point geometry.

PROBLEM 9.11. Each pair of gymnastic meets that happens in a given week provides a pair of parallel lines. Therefore we get $\binom{3}{2} = 3$ pairs of parallel lines for each week of the season.

For instance, here are the three pairs of parallel lines corresponding to the first week:
$\{A,B,C\}$ and $\{D,E,F\}$; $\{A,B,C\}$ and $\{G,H.I\}$; $\{D,E,F\}$ and $\{G,H.I\}$.

Altogether, this process leads to $4 \cdot 3 = 12$ pairs of parallel lines.

Now, are these all of the parallel pairs? I claim that they are. Here's a counting argument that verifies my claim. Since there are 12 lines altogether, there are $\binom{12}{2} = 66$ pairs of distinct lines. How many ***non***-parallel pairs are there? Well, each team competes in one meet each week for a total of 4 meets. This means that there are $\binom{4}{2} = 6$ pairs of lines in our geometry which have a given point as the intersection of the pair. For instance, just for clarity [since you're ***not*** required to provide such a list], let's observe that Team A participates in the following 4 meets:
$\{A,B,C\}$, $\{A,D,G\}$, $\{A,E,I\}$, $\{A,F,H\}$;
and therefore we get the following **6 *pairs of lines*** in our geometry that intersect at A:
$\{A,B,C\}$ and $\{A,D,G\}$; $\{A,B,C\}$ and $\{A,E,I\}$; $\{A,B,C\}$ and $\{A,F,H\}$;
$\{A,D,G\}$ and $\{A,E,I\}$; $\{A,D,G\}$ and $\{A,F,H\}$; $\{A,E,I\}$ and $\{A,F,H\}$.

Now finally, since there are 6 intersecting pairs of lines at each of the 9 points, we have a total of $9 \cdot 6 = 54$ pairs of intersecting lines. (And indeed, the uniqueness part of axiom I1 guarantees that there are no overlaps among the 9 lists.) Since we have accounted for 12 parallel pairs and 54 intersecting pairs, we have accounted for all $12 + 54 = 66$ pairs of lines. This shows that there are ***no parallel pairs other than the ones that we found***, as claimed.

Note: Once again we have a finite situation, so that we could just enumerate everything. But clearly, that would be excessively tedious. On the other hand there ***very well could be*** other valid ways to do the problem in an organized fashion.

PROBLEM 9.22. We defined between-ness by saying that $B = (b_1, b_2)$ is between $A = (a_1, a_2)$ and $C = (c_1, c_2)$ if and only if one of the inequalities $a_1 < b_1 < c_1$ or $a_1 > b_1 > c_1$ holds. Now, the difficulty that would be involved with using the y-coordinate is that it does not vary monotonically with the x-coordinate as we move across the half circle. Equivalently, we can say that even if the x-coordinates of the three points are linearly ordered as above, the y-coordinates can fail to be linearly ordered. The clearest failure of this can be exhibited by taking B to be the point at the top of the half circle. Indeed, in this situation, the ordering $a_1 < b_1 < c_1$ corresponds to having A on the left side and C on the right side. But if B is at the top of the circle, we'll have $a_2 < b_2$ and also $b_2 > c_2$. A similar conclusion holds if the positions of A and C are reversed. It follows that the y-coordinates ***do not have*** the necessary linear ordering.

Back to the class homepage.