Math 5385 Fall 2008

Final Project/Take-home exam
Due Tuesday, December 16

Work any ***three*** problems.

I. ***Topic:*** dealing with denominators in a parametrization. This is based on Exercise 12 in Sec 3.3 of the text. Thus, we consider the parametric surface $x = \frac{u^2}{v}$, $y = \frac{v^2}{u}$, $z = u$, and we start by working with the ideal $I = \langle vx-u^2, uy-v^2, z-u\rangle$ in $k[u,v,x,y,z]$.

a) Verify that every point (x,y,z) in the image of this parametrization is contained in the variety $V(x^2y - z^3)$.

b) Let I_2 be the second elimination ideal $I \cap k[x,y,z]$. Verify that $I_2 = \langle z(x^2y - z^3)\rangle$.
Note: There is a misprint in the statement of this part of the problem in the text.

c) Show that $V(I_2)$ is ***not*** the smallest variety that contains the image of this parametrization.

d) Let $g = uv$, the product of the denominators in the parametrization. We introduce an extra variable w, which is to be subject to the relation $wg = 1$, in order to deal with the problem created by vanishing denominators. Thus, instead of working with I, we now work with the ideal $J = \langle vx-u^2, uy-v^2, z-u, uvw-1\rangle$ in $k[u,v,w,x,y,z]$.

e) Let J_3 be the third elimination ideal $J \cap k[x,y,z]$. Does this answer correctly tell us what is the smallest variety that contains the image of the parametrization? Please explain briefly.

II. ***Topic:*** finding the singular points in a particular curve.
In this problem, we take $f = 5(x^2-1)(y^2-1) - 2(x^2-1)^4 - 2(y^2-1)^4$.

a) Find the singular points of the plane curve $V(f)$.
Suggestion: you can begin by finding a Groebner basis of the ideal $\left\langle f, \partial f/\partial x, \partial f/\partial y \right\rangle$ relative to lex order.

b) Use the Matlab script `aPlot` (or a comparable method) to draw a picture showing the curve, with a grid that includes the singular points.

c) Use the Matlab script `fCone` to draw a picture that shows the curve and the tangent cones at the various singular points.

III. ***Topic:*** Finding the singular curves in a family of curves.

a) = Exercise 20 in sec 3.4 (page 150 of the text)
Suggestion: You can do this by finding a Groebner basis relative to lex order, but you must make a correct choice of which variable is to be last.

b) Apply the method that you found in part a) to find the singular curves in the following specific family, and then find the singular points on each singular curve. (This is the same family that was considered in Matlab exercise 3.
➢ $y^2 = x^3 - 3x + t$

c) Review your sketches from Matlab Exercise 3 (or re-do it if necessary), and re-evaluate it by comparing with the answer from part b) above, to determine which point or points of any curve was (or were) skipped by the Matlab sketch. If there was such a point, what point was it, and which curve did it belong to?

d) (Optional): Animate the family, again using an appropriate Matlab script. Save the movie as a series of files in a subdirectory. (This can be done using the script `writeMovie`. Other relevant scripts are `makeMovie`, `readMovie`, and `playMovie`.)

IV. ***Surface plotting*** **(A more detailed help sheet will be handed out on Wednesday)**

The goal is to plot a surface given parametrically by $x = f(u,v)$, $y = g(u,v)$, $z = h(u,v)$. If using Matlab, we begin by entering the function: if $f(u,v) = uv$, $g(u,v) = u^2$, and $h(u,v) = v^2$, for instance, then we enter something like the following:

```
f = 'u*v'        (The single quotes are important.)
g='u^2'
h='v^2'
F=vectorize(f)
G=vectorize(g)
H=vectorize(h)
```

And then we enter the gridpoints for the parameter functions. For instance, if u and v are both to go from 0 to 1 in steps of 0.01, we could enter:

```
r = 0:0.01:1;        (Use the semicolon to avoid filling the computer screen.)
s = 0:0.01:1;
[u,v] = meshgrid(r,s);
```

Now we evaluate the functions at the gridpoints:

```
X = eval(F);
Y = eval(G);
Z = eval(H);
```

And finally we can plot the surface:

```
S = surf(X,Y,Z);
```

a) Plot the tangent surface of the twisted cubic, as given parametrically on page 128 of the text, with the parameter ranges $-1 \le t \le 1$, $-1 \le u \le 1$. To show the tangent line structure most effectively, try just allowing the three parameter values -1, 0, 1 in the u-direction. (But use an ample number of parameter values in the t-direction.

b) What are the lengths of the tangent line segments in the plot which correspond to $t = 0$ and $t = 1$ respectively?

c) (Optional) Is there a way to modify the parametrization to make the tangent line segments all have equal lengths?

d) Plot the surface described on page 16 of the text.

e) (Optional) Use the Maple command `implicitplot3d` (part of the `plots` package in Maple) to plot the surface $V(x^2 - y^2z^2 + z^3)$ in the range $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$. {According to sec 1.2 of the text, this is the same as the surface in part d).} Compare this plot to the plot from part d).

V. ***Topic:*** The Closure Theorem.

a) Describe the image of the surface $z = xy$ under the projection to the yz-plane. In particular, which points on the y-axis actually are in the image? (You may find that ***a lot of*** points are omitted, but you should find that something is not omitted.)

b) Does the material in part a) constitute a valid example of the closure theorem? Please explain.

c) We consider the ideal $I = \langle wx - y,\ wy - z \rangle$ in $k[w,x,y,z]$. Let I_1 be the first elimination ideal $I \cap k[x,y,z]$. Find a Groebner basis for I_1. Describe the variety W mentioned in the Closure Theorem, and determine whether or not the inclusion $V(I_1) - W \subset \pi_1(V(I))$ is strict.