7622 Complex Eigenvalues

Example 1. Find the eigenvalues and corresponding eigenvectors for the matrix

$$B = \begin{pmatrix} 7 & -5 \\ 13 & -1 \end{pmatrix}.$$

Solution. Recall the definition of eigenvalue and corresponding eigenvector. The scalar λ is an eigenvalue and the vector $\vec{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is the corresponding eigenvector of the matrix B if

$$\begin{pmatrix} 7 & -5 \\ 13 & -1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \lambda \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$$

This equation can be rewritten as

$$\begin{pmatrix} 7-\lambda & -5 \\ 13 & -1-\lambda \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.1}$$

This is a vector equation. It is equivalent to a system of two equations in the unknowns V_1 and V_2. This homogeneous system has non zero solutions if the determinant of the coefficient matrix is zero. The scalar λ is an eigenvalue of the matrix B if λ is chosen such that

$$\begin{vmatrix} 7-\lambda & -5 \\ 13 & -1-\lambda \end{vmatrix} = 0.$$

$$\begin{aligned} (7-\lambda)(-1-\lambda) + 65 &= 0 \\ -7 - 7\lambda + \lambda + \lambda^2 + 65 &= 0 \\ \lambda^2 - 6\lambda + 58 &= 0. \end{aligned}$$

We cannot solve this equation by factoring. We could solve it by using the quadratic formula. We will solve it by completing the square. We rewrite the equation as

$$\begin{aligned} \lambda^2 - 6\lambda + 9 + 49 &= 0 \\ (\lambda - 3)^2 &= -49 \\ \lambda - 3 &= \pm 7i. \end{aligned}$$

The solutions are $\lambda = 3 + 7i$ and $\lambda = 3 - 7i$.

The eigenvalues are $3+7i$ and $3-7i$. The numbers $3+7i$ and $3-7i$ are called conjugates of one another. Since the eigenvalues are complex numbers some of the components of the eigenvectors will also be complex numbers.

Replacing λ with the eigenvalue $3+7i$ in the vector equation (2.1), we get

$$\begin{pmatrix} 7-3-7i & -5 \\ 13 & -1-3-7i \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

or

$$\begin{pmatrix} 4-7i & -5 \\ 13 & -4-7i \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is the same as a system of two equations in two unknowns with infinitely many solutions. Even though the coefficients of the unknowns are complex we solve the system of equations using the same steps as if the numbers were real. When we had real numbers we could usually see right away that the top and bottom equations are actually the same equation. For this system it is not at all clear that the bottom equation is a multiple of the top equation. However, we chose λ to be $3+7i$ in order to make this a dependent system. The bottom equation is a multiple of the top equation. The bottom is $(4/5)+(7/5)i$ times the top equation. For the moment, we just assume that this is true. Let us look at the top equation

$$(4-7i)V_1 - 5V_2 = 0,$$

and find all the solutions of this equation. The easiest solution is to take $V_1 = 5$ and $V_2 = 4-7i$. Substituting these values for V_1 and V_2 we easily see that these values give a solution:

$$(4-7i)(5) - 5(4-7i) = 0.$$

In order to make sure that we have not made a mistake we check $V_1 = 5$ and $V_2 = 4-7i$ in the bottom equation.

$$\begin{aligned} 13(5) + (-4-7i)(4-7i) &= 0 \\ 65 - 16 + 28i - 28i - 49 &= 0 \\ 0 &= 0. \end{aligned}$$

These values do check. Thus $V_1 = 5$ and $V_2 = 4 - 7i$ is a solution of the bottom equation. An eigenvector of the matrix B corresponding to $\lambda = 3 + 7i$ is the vector

$$\begin{pmatrix} 5 \\ 4 - 7i \end{pmatrix}.$$

Recall that we can multiply this vector by a scalar and the result is also an eigenvector. All the vectors of the form

$$s\begin{pmatrix} 5 \\ 4 - 7i \end{pmatrix}$$

are eigenvectors of B corresponding to $\lambda = 3 + 7i$. In this case we assume the scalar s can be any complex number. For example, if $s = 3i$, we get

$$\begin{pmatrix} 15i \\ 21 + 12i \end{pmatrix}$$

as an eigenvector.

The other eigenvalue is $\lambda = 3 - 7i$. The number $3 - 7i$ is the conjugate of $3 + 7i$. Substituting $3 - 7i$ for λ in the vector equation (1) for V_1 and V_2, we get

$$\begin{pmatrix} 7 - 3 + 7i & -5 \\ 13 & -1 - 3 + 7i \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 4 + 7i & -5 \\ 13 & -4 + 7i \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This vector equation is equivalent to the two simultaneous equations in two unknowns. These two equations are dependent. The bottom equation is a multiple of the top equation. The top equation is

$$(4 + 7i)V_1 - 5V_2 = 0.$$

The easy solution of this equation is to let $V_1 = 5$ and $V_2 = 4 + 7i$. These two values clearly satisfy this equation:

$$(4 + 7i)(5) - 5(4 + 7i) = 0.$$

We also need to make sure that $V_1 = 5$ and $V_2 = 4 + 7i$ is also a solution of the second equation. Substituting these values into the second equation, we get

$$\begin{aligned} 13(5) + (-4 + 7i)(4 + 7i) &= 0 \\ 65 - 16 - 28i + 28i - 49 &= 0 \\ 0 &= 0. \end{aligned}$$

Thus $V_1 = 5,\ V_2 = 4 + 7i$ is a solution of both equations. The eigenvector of B corresponding to $\lambda = 3 - 7i$ is

$$\begin{pmatrix} 5 \\ 4 + 7i \end{pmatrix}.$$

In fact all the vectors of the form

$$s \begin{pmatrix} 5 \\ 4 + 7i \end{pmatrix} = \begin{pmatrix} 5s \\ [4 + 7i]s \end{pmatrix}$$

where s is a complex number are actually eigenvectors of B corresponding to the eigenvalue $\lambda = 3 - 7i$. We can show this by substituting $V_1 = 5s$ and $V_2 = [4 + 7i]s$ into the vector equation (2.1):

$$\begin{pmatrix} 7 - 3 + 7i & -5 \\ 13 & -1 - 3 + 7i \end{pmatrix} \begin{pmatrix} 5s \\ [4 + 7i]s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Multiplying, we get

$$\begin{pmatrix} 20s + 35si - 20s - 35si \\ 65s - 16s - 28si + 28si - 49s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is clearly true and so the value $V_1 = 5s$ and $V_2 = [4 + 7i]s$ do check in the defining system of equations.

Let us look again at the original equation that defined the eigenvalue and corresponding eigenvector of B. We already said that according to this equation the scalar $\lambda = 3 - 7i$ is an eigenvalue of matrix B and the vector $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is the corresponding eigenvector if

$$\begin{pmatrix} 4 + 7i & -5 \\ 13 & -4 + 7i \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is the same equation we considered above. We already found the eigenvector which is the solution of this equation by starting with the top equation. We are going to do a different solution of this vector equation. Let us do the solution of this system of equations again, but this time let us start with the bottom equation first. The bottom equation is

$$13V_1 + (-4 + 7i)V_2 = 0.$$

The easy solution of the bottom equation is $V_1 = 4 - 7i$ and $V_2 = 13$. This is clearly a solution since substitution gives

$$13(4 - 7i) - (4 - 7i)(13) = 0.$$

These values of V_1 and V_2 must be a solution of the first equation. Substituting $V_1 = 4 - 7i$ and $V_2 = 13$ into the first equation, we get

$$\begin{aligned}(4 + 7i)(4 - 7i) - 5(13) = 0 \\ 16 + 49 - 65 = 0.\end{aligned}$$

These values check in both equations. Therefore, we can write the eigenvector corresponding to $\lambda = 3 - 7i$ as

$$\begin{pmatrix} 4 - 7i \\ 13 \end{pmatrix}.$$

The first time when we found this eigenvector we wrote it as

$$\begin{pmatrix} 5 \\ 4 + 7i \end{pmatrix}.$$

At first these two expressions do not look like this same eigenvector. But let us multiply this first form of the vector by the scalar $(1/5)(4 - 7i)$, we get

$$(1/5)(4 - 7i)\begin{pmatrix} 5 \\ 4 + 7i \end{pmatrix} = (1/5)\begin{pmatrix} 5(4 - 7i) \\ [4 - 7i][4 + 7i] \end{pmatrix} =$$
$$(1/5)\begin{pmatrix} 5[4 - 7i] \\ 65 \end{pmatrix} = \begin{pmatrix} 4 - 7i \\ 13 \end{pmatrix}.$$

This is the second eigenvector. The second way of writing the vector $\begin{pmatrix} 4-7i \\ 13 \end{pmatrix}$ is just a scalar, namely $(1/5)(4-7i)$, times the first way of writing the vector which was $\begin{pmatrix} 5 \\ 4+7i \end{pmatrix}$.

It is time to notice another very interesting fact about eigenvectors: The two eigenvalues $3+2i$ and $3-7i$ are conjugates of one another. The eigenvector corresponding to $\lambda = 3+7i$ is

$$\begin{pmatrix} 5 \\ 4-7i \end{pmatrix}.$$

The eigenvector corresponding to $\lambda = 3-7i$ is

$$\begin{pmatrix} 5 \\ 4+7i \end{pmatrix}.$$

Note that the eigenvectors are also conjugates of one another. As we work more problems we see that the complex eigenvalues of a real matrix always come in conjugate pairs. This is a good thing because it follows that the corresponding eigenvectors are also conjugates. This means that as soon as we find the eigenvector for $\lambda = 3+7i$ we can automatically write down the vector for $\lambda = 3-7i$.

Exercises

Find the eigenvalues and corresponding eigenvectors for the following matrices.

1. $\begin{pmatrix} 10 & -5 \\ 8 & -2 \end{pmatrix}$
2. $\begin{pmatrix} 7 & 13 \\ -1 & 3 \end{pmatrix}$
3. $\begin{pmatrix} 5 & -17 \\ 2 & -1 \end{pmatrix}$
4. $\begin{pmatrix} 10 & -5 \\ 5 & 4 \end{pmatrix}$
5. $\begin{pmatrix} 7 & -13 \\ 5 & -1 \end{pmatrix}$