As always, the review sheets in workshop are a great way to review for the exam. They're written by the same person who writes the exam, so you can get a good feel for what will be emphasized, and what the problems will look like. (Not surprisingly, they're similar to the problems in the lecture notes and homework.)

However, some of you won't get those review sheets until Thursday, mere hours before the exam. We'll also review for the exam on Wednesday in class, but that's not enough time to go over everything. So for those of you who would like to start studying early, here are some guidelines.

Instead of section-by-section, I'm organizing this by topic.

Update: (Wednesday afternoon) I've fixed a few typos and added the list formulas which will be on the front page of the exam.

Homogeneous Systems of Differential Equations

The problem here is to solve a system like

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We do this using eigenvalues and eigenvectors. Suppose the eigenvalues of the coefficient matrix are $\lambda = 1$ and $\lambda = -3$, corresponding to the eigenvectors $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -7 \\ 9 \end{pmatrix}$, respectively. Then

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{1t} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = e^{-3t} \begin{pmatrix} -7 \\ 9 \end{pmatrix}$$

are both solutions to the system of equations. The general solution is the set of all possible combinations:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{1t} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} -7 \\ 9 \end{pmatrix}$$

Warning: things get a little tricky if the eigenvalues and eigenvectors are complex. If we have an eigenvalue $\lambda = 2+3i$ with eigenvector $\begin{pmatrix} 1 \\ 6-5i \end{pmatrix}$ in a two by two system, then you should know how to find the other eigenvalue and its corresponding vector. (Conjugate the value and vector I just gave you.) And it's still true that

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{(2+3i)t} \begin{pmatrix} 1 \\ 6-5i \end{pmatrix} + C_2 e^{(2-3i)t} \begin{pmatrix} 1 \\ 6+5i \end{pmatrix}$$

is the general solution, but we usually require the answer to be written with real numbers. Strangely enough, you can do this using just *one* of the eigenvalues and eigenvectors. Let

$$y_1 = C_1 e^{(2+3i)t} \begin{pmatrix} 1 \\ 6-5i \end{pmatrix}$$

Then the general solution is

$$y = C_1(\text{Real part of } y_1) + C_1(\text{Imaginary part of } y_1)$$

You can find out how to compute the real and imaginary parts in your lecture notes.

Nonhomogeneous Systems of Differential Equations

A nonhomogenous system has the form $\mathbf{x}' = B\mathbf{x} + \mathbf{f(t)}$, as in

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + e^{-5t} \begin{pmatrix} 45 \\ 33 \end{pmatrix}$$

or

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \cos 4t \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

We solve these by writing down the general solution to the associated homogeneous system (where you lop off the last term), and then you add a particular solution. The particular solution is found using the method of undetermined coefficients. It's very similar to what we did on the last exam, except now you have to make "two guesses at once." For example, the particular guesses for the above systems would be

$$\begin{pmatrix} x_p \\ y_p \end{pmatrix} + e^{-5t} \begin{pmatrix} A \\ B \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_p \\ y_p \end{pmatrix} + \cos 4t \begin{pmatrix} A \\ B \end{pmatrix}$$

Other Topics with Systems of Equations

There are a few other things you should know how to do with systems of equations. Although I haven't included pages and pages describing each of these problems, any one of them could be worth 10-20 points. Make sure you can do them.

Brine Problems: You should be able to take the description of an interconnected pair of brine tanks and set up a system of equations. (Using the techniques described earlier, you should know how to solve the system, too.) If you can do both of the homework problems from section 7782 you should be fine.

Systems in non-standard form: You should be able to take a system like

$$\begin{aligned} x' + 3y' - 4x + 3y &= 19 \\ 2x' - y' + x - 7y &= 23 \end{aligned}$$

and write it in our standard form, so that it can be solved using the methods described earlier.

Uncoupling: *You have to know how to use a substitution to "uncouple" a system.* This is in section 7774 of your text. It involves diagonalizing the coefficient matrix in the system.

Laplace Transforms and Inverse Laplace Transforms

It doesn't take a lot of space to describe the problems you'll have to do involving the Laplace Transform and its inverse, but these problems are time consuming; make sure you've practiced these types of problems until you can do them quickly!

You should be able to do the following.

- Solve first and second order initial value problems using $\mathcal{L}$ and $\mathcal{L}^{-1}$.
- In particular, computing $\mathcal{L}^{-1}$ usually involves some partial fractions. You won't have to deal with any repeated terms, though; just distinct linear and irreducible quadratic factors. In other words, you might have a denominator of $s(s-4)$ or $(s-4)(s^2+9)$, but not $s(s-3)^3$ or $(s-2)(s^2+4)^2$.
- To save yourself a lot of trouble, recognize when the partial fractions decomposition has already been done. For example, if presented with

$$\mathcal{L}^{-1}\left\{\frac{2s-9}{s^2+6s+34}\right\}$$

you should recognize that the denominator can't be factored, but the top is already of the form $As+B$, which is what you'd get from the whole partial fractions process. So you shouldn't try partial fractions at all. Instead you should rewrite it by completing the square:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2s-9}{s^2+6s+34}\right\} &= \mathcal{L}^{-1}\left\{\frac{2s-9}{s^2+6s+9-9+34}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{2s-9}{(s+3)^2+25}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{2(s+3)-6-9}{(s+3)^2+25}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{2(s+3)-15}{(s+3)^2+25}\right\}\\
&= \mathcal{L}^{-1}\left\{2\frac{(s+3)}{(s+3)^2+25}-3\frac{5}{(s+3)^2+25}\right\}\\
&= 2e^{-3t}\cos(5t)-3e^{-3t}\sin(5t)
\end{aligned}$$

To write down the answer at the end I used the inverse of the First Translation Theorem. That can be tricky, so if you're fuzzy on those details you should definitely review it.
- In addition to the First Translation Theorem and its inverse, you need to be comfortable with the Second Translation Theorem and its inverse. For example, you should be able to compute:

$$\mathcal{L}^{-1}\left\{e^{-3s}\left[\frac{7}{s^2}-\frac{5}{s+2}\right]\right\}$$

- The inverse of the Second Translation Theorem results in a term with the Heaviside function. You should be able to take all of these terms and figure out the resulting piecewise function, and sketch its graph. Working through examples 4 and 5 in section 7731 would be good preparation for this.

Formulas

You'll be given the formulas for:

- $\mathcal{L}\{t^n\}$
- $\mathcal{L}\{e^{at}\}$
- $\mathcal{L}\{\sin bt\}$
- $\mathcal{L}\{\cos bt\}$
- $\mathcal{L}\{f'(t)\}$
- $\mathcal{L}\{f''(t)\}$
- $\mathcal{L}\{e^{at}f(t)\}$
- $\mathcal{L}\{f(t-c)H(t-c)\}$