Math 3283W - Exam 2 Review Worksheet[1]

November 5, 2012

§7 - Functions

1. For a function $f : A \to B$, the function $g : B \to A$ is called

   a **left inverse** for f if $g \circ f$ is the identity on A (i.e., $g \circ f = \mathrm{id}_A$)

   and

   a **right inverse** for f if $f \circ g$ is the identity on B (i.e., $f \circ g = \mathrm{id}_B$).

   (a) Prove: f has a left inverse if and only if f is injective.

   (b) Prove: f has a right inverse if and only if f is surjective.

2. Let A, B, C be sets such that $C \subseteq B$. Prove that $f^{-1}(B \setminus C) = A \setminus f^{-1}(C)$.

3. Let A, B, C be sets such that $C \subseteq B$.

   (a) Prove: if f is surjective, then $f(f^{-1}(C)) = C$.

   (b) Give an example of an function f such that $f(f^{-1}(C)) \subsetneq C$.

§8 - Cardinality

1. For S and T sets, state the definition of $|S| \le |T|$.

2. Give an example of a set $S \subsetneq \mathbb{R}$ such that

   (a) S is denumerable.

   (b) S is uncountable.

   Prove your claims. You may use the fact that $\mathbb{R}$ is uncountable.

3. Recall, the power set $\mathcal{P}(S)$ of a set S is defined by the property

   $$A \in \mathcal{P}(S) \iff A \subseteq S.$$

   (a) What is $\mathcal{P}(\{a, b, c, d\})$?

   (b) Prove: for every set S, $|S| \le |\mathcal{P}(S)|$. (In fact, $|S| < |\mathcal{P}(S)|$, though you are not being asked to prove this stronger statement.)

4. Prove: if S is countable, then there exists a proper subset T of S such that $S \sim T$.

§10 - Induction

[1]This worksheet is far from inclusive. Do **not** assume that doing only these problems will fully prepare you for Exam 2.

1. Prove using induction: $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$.

2. Prove using induction: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{N}$.

3. Prove using (1) and (2): $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ for all $n \in \mathbb{N}$.

§11 - Fields

1. Prove that $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$.

2. Prove that $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

§12 - Completeness

1. State the definition of supremum and infimum for a set S.

2. For each set listed below, determine whether the supremum and/or infimum exist. If so, determine the value of the supremum and/or infimum. If not, state why.

   (a) $\{\frac{n}{n+1} : n \in \mathbb{N}\}$
   (b) $(-\infty, 4)$
   (c) $\{\frac{1}{n} : n \in \mathbb{N}\}$

3. Let S and T be bounded, nonempty sets such that $S \subseteq T \subseteq \mathbb{R}$. Prove

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

§13 - Topology

1. Consider the following sets:

   (a) $\{\pi\}$
   (b) $\mathbb{Q}$
   (c) $\bigcap_{n=1}^{\infty}(0, \frac{1}{n})$
   (d) $(x, y]$ where $x < y$ are real numbers.

   For each set, determine and prove:

   - Is the set open, closed, both, or neither?
   - The boundary and interior of the set.

2. (a) Prove: $\mathrm{cl}\,(S \cap T) \subseteq \mathrm{cl}\,(S) \cap \mathrm{cl}\,(T)$ (*Hint:* Use the characterization of $\mathrm{cl}\,(S)$ that $x \in \mathrm{cl}\,(S)$ if and only if every neighborhood of x intersects S non-trivially.)
   (b) Give an example of S and T such that $\mathrm{cl}\,(S \cap T) \subsetneq \mathrm{cl}\,(S) \cap \mathrm{cl}\,(T)$.

3. Let $S \subseteq \mathbb{R}$ be nonempty, open, and bounded above. Prove that $\sup S \notin U$.