The following is a non-comprehensive list of solutions to the skills problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

17.3(a) We can evaluate this limit as follows:

$$\begin{aligned}
\lim \frac{3n^2+4n}{7n^2-5n} &= \lim \frac{3+\frac{4}{n}}{7-\frac{5}{n}} && \text{(Algebra)}\\
&= \frac{\lim\left(3+\frac{4}{n}\right)}{\lim\left(7-\frac{5}{n}\right)} && \text{(Thm 17.1d)}\\
&= \frac{\lim 3+\lim 4/n}{\lim 7-\lim 5/n} && \text{(Thm 17.1a)}\\
&= \frac{\lim 3+4\lim 1/n}{\lim 7-5\lim 1/n} && \text{(Thm 17.1b)}\\
&= \frac{3+4\cdot 0}{7-5\cdot 0}\\
&= \frac{3}{7}
\end{aligned}$$

17.6 (a) $s_n = n$ and $t_n = -n$ work as a counterexample, since they diverge but $s_n + t_n = 0$ for all $n$, and $(0, 0, 0, 0, \ldots)$ converges.

(b) Choosing $s_n$ and $t_n$ to both equal $(-1)^n$ works as a counterexample.

(c) This is true, using Theorem 17.1, but you have to be careful about only applying Theorem 17.1 when it applies. First note that $t_n = s_n + t_n - s_n$. Then:

$$\begin{aligned}
\lim t_n &= \lim\left[s_n+t_n-s_n\right]\\
&= \lim\left[(s_n+t_n)-s_n\right]\\
&= \lim(s_n+t_n)-\lim t_n
\end{aligned}$$

We can write that last line using Theorem 17.1, because we're told that $\lim s_n + t_n$ and $\lim t_n$ both exist. Hence $\lim t_n$ also exists and equals $(\lim s_n + t_n) - (\lim t_n)$.

(d) $s_n = \dfrac{1}{n^2}$ and $t_n = n$ provide a counterexample to this statement.

17.8 (a) There are lots of possible answers. The simplest might be the constant sequence $(s_n) = (1, 1, 1, 1, 1, 1, 1, \ldots.)$. But $s_n = \dfrac{1}{n}$ and other choices would work, too.

(b) $t_n = n$ provides a nice, simple example for this statement.

17.15(c) We can evaluate this limit as follows:

$$\begin{aligned}
\lim\left(\sqrt{n^2+n}-n\right) &= \lim \frac{\sqrt{n^2+n}-n}{1}\cdot\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\\
&= \lim \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}\\
&= \lim \frac{n}{\sqrt{n^2+n}+n}\\
&= \lim \frac{1}{\sqrt{1+\frac{1}{n}}+1}\\
&= \frac{1}{1+1}\\
&= \frac{1}{2}
\end{aligned}$$

18.3 For both (a) and (b) you need to do two induction proofs, one to show the sequence is bounded, and one to show it is monotone. That's enough to say (via the Monotone Convergence Theorem) that the sequence converges. Then you make use of the general fact that, for any convergent sequence,

$$\lim s_{n+1} = \lim s_n$$

to find the actual limit.

(a) After looking at a few values, we suspect $s_n$ is increasing and never reaches 2. Let's first prove 2 is an upper bound:

**Base case**: $s_1 = 1 < 2$.

**Induction step**: Assume $s_k < 2$. Then

$$s_{k+1} = \frac{1}{4}(s_n+5) < \frac{1}{4}(2+5) = \frac{7}{4} < 2$$

Hence $s_{k+1}$ is less than 2 as well. This completes the proof by induction that $s_n < 2$ for all $n$.

Next we prove that $s_n$ is increasing:

**Base case**: $s_1 = 1 < \frac{3}{2} = s_2$.

**Induction Step**: Now assume $s_{k-1} < s_k$ and use that to prove $s_k < s_{k+1}$. Using our assumption and the definition of $s_n$,

$$s_{k+1} = \frac{1}{4}(s_k+5) > \frac{1}{4}(s_{k-1}+5) = s_k$$

Hence $s_k < s_{k+1}$ as desired, completing our proof.

Thus $s_n$ converges to some $s$ by the Monotone Convergence Theorem. It's limit can be found as follows.

$$\begin{aligned}\lim s_{n+1} &= \lim s_n\\ \lim \frac{1}{4}(s_n+5) &= \lim s_n\\ \frac{1}{4}(s+5) &= s\\ s &= \frac{5}{3}\end{aligned}$$

(d) This proof follows the same framework as (a), so I won't write out all of the details. By examining a few values, we suspect that $s_n$ is increasing and never reaches 3. It's certainly true that $s_1 < 3$ and $s_1 < s_2$, the base cases for our two inductive proofs. To complete the proof that $s_n < 3$ for all $n$, assume $s_k < 3$ and note that

$$s_{k+1} = \sqrt{2s_k+1} < \sqrt{2\cdot 3+1} = \sqrt{7} < 3$$

For the inductive proof that $s_n$ is increasing, assume $s_k > s_{k-1}$ and check the next element in the sequence:

$$s_{k+1} = \sqrt{2s_k+1} > \sqrt{2s_{k-1}+1} = s_k$$

The limit of the sequence can be found by setting

$$\begin{aligned}\lim s_{n+1} &= \lim s_n\\ \lim \sqrt{2s_n+1} &= \lim s_n\\ \sqrt{2s+1} &= s\end{aligned}$$

Which gives $s = 1+\sqrt{2}$.

18.4 In this problem it's helpful to remember that *Cauchy* and *convergent* are equivalent.

(a) $s_n = \dfrac{(-1)^n}{n}$ works; it converges, but the alternating sign means it's neither increasing nor decreasing.

(b) $s_n = n$ works.

(c) $s_n = (-1)^n$ works. It's bounded–the values in the sequence are just 1 and $-1$–but it does not converge.