The following is a non-comprehensive list of solutions to the skills problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

16.7 For the midterm you should aim to be able to prove any limits like (a)-(d) using algebra and the limit laws (Theorem 17.1); ask us if you're not sure how. Sometimes tools like Theorem 17.7 can be useful, too. For example,

(e) With $s_n = \frac{n^2}{n!}$, we have $\frac{s_{n+1}}{s_n} = \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} = \frac{(n+1)^2}{(n+1)n^2} = \frac{n^2+2n+1}{n^3+n^2}$ which converges to 0. Hence $s_n \to 0$ by Theorem 17.7.

(f) We break this into cases; if $x = 0$ then $(s_n) = (x^n) = (0, 0, 0, \ldots)$, a constant sequence which certainly converges. If $0 < x < 1$, then

$$\frac{s_{n+1}}{s_n} = \frac{x^{n+1}}{x^n} = x < 1$$

So $s_n \to 0$ by Theorem 17.7.

If $-1 < x < 0$ then we again have $\frac{s_{n+1}}{s_n} = x$, but the problem is that Theorem 17.7 does not directly apply, since our sequence $(s_n) = (x^n)$ is not a sequence of positive terms. However, the sequence $(|x|^n)$ converges to 0 (by the previous paragraph), and it is the case that if $|s_n| \to 0$, then $s_n \to 0$. This is actually part (c) of Exercise 16.9, which was not assigned, but is fairly quick to prove. The key idea is that

$$s_n \text{ is within } \epsilon \text{ of } 0 \;\Leftrightarrow\; |s_n| \text{ is within } \epsilon \text{ of } 0$$

that is,

$$|s_n - 0| = |s_n| < \epsilon \Leftrightarrow ||s_n| - 0| = |s_n| < \epsilon$$

18.10 (a) You proved in Exercise 10.7 that $(1 + r + r^2 + \cdots + r^n) = \frac{1 - r^{n+1}}{1 - r}$. In Exercise 16.7(f) we saw that $\lim x^n = 0$ for $|x| < 1$. Hence if $|r| < 1$, we have:

$$\lim 1 + r + r^2 + \cdots + r^n = \lim \frac{1 - r^{n+1}}{1 - r} = \frac{1 - 0}{1 - r} = \frac{1}{1 - r}$$

(b) Using the previous part and the fact that $\sum ca_n = c \sum a_n$, this is the following geometric series:

$$\sum_{n=0} \left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^n = \frac{9}{10}\sum_{n=0}^{\infty}\left(\frac{1}{10}\right)^n = \left(\frac{9}{10}\right)\frac{1}{1 - 1/10} = \cdots = 1$$

32.5(a) **Updated**: This series sums to 1/2. It's a geometric series with $r = 1/3$, minus the first term.

32.5(b) **Updated**: This series sums to 1/4. It's a geometric series with $r = 1/2$, minus the first three terms.

32.5(c) This series sums to 4/3. It's a geometric series with $r = -1/2$ and $a = 2$.

32.5(e) This series sums to 1. It's a telescoping series, because[1]

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

which means (remember that n starts at 2):

$$\begin{aligned} s_2 &= \frac{1}{1} - \frac{1}{2} \\ s_3 &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} \\ s_4 &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \end{aligned}$$

Noticing the pattern, we see most things cancel, and the n^{th} partial sum is

$$s_n = 1 - \frac{1}{n}.$$

Since $s_n \to 1 - 0 = 1$, the series converges to 1.

32.5(g) This series sums to 1/3. It's another telescoping series, with

$$\frac{1}{(4n-2)(3n+1)} = \frac{1/3}{3n-2} - \frac{1/3}{3n+1}$$

which leads to the following formula for the n^{th} partial sum:

$$s_n = \frac{1}{3} - \frac{1/3}{3n+1}$$

which converges to 1/3.

32.5(h) This is another telescoping series and sums to $\frac{3}{2}$. In this series,

$$\frac{2}{n^2+2n} = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$

The "telescoping" doesn't happen right away in this series:

$$\begin{aligned} s_1 &= \frac{1}{1} - \frac{1}{3} \\ s_2 &= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} \\ s_3 &= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} \\ s_4 &= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} \\ s_5 &= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} \\ s_6 &= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} \end{aligned}$$

There are a *lot* of numbers here, but hopefully you catch the pattern: every time a fraction (like $\frac{1}{3}$) is subtracted, it is cancelled out by the addition of the same fraction, three numbers later. The only numbers which are never cancelled are the initial $\frac{1}{1}$ and $\frac{1}{2}$, which is the intuitive reason for why the series sums to $1 + 1/2 = 3/2$. If you write out a formula for s_n you'll also see it converges to $3/2$ as $n \to \infty$.

[1] Talk to us in office hours or in class on Tuesday/Thursday if you need practice figuring out how to "split" these fractions apart.

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