The following is a non-comprehensive list of solutions to the skills problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

6.11 (e)
- It is reflexive. For any x, we have $x - x = 0 \in \mathbb{Q}$.
- It is symmetric. If $x - y = \frac{p}{q} \in \mathbb{Q}$, then $y - x = -\frac{p}{q} \in \mathbb{Q}$.
- It is transitive. This requires a bit more work: if $x - y = \frac{p}{q} \in \mathbb{Q}$ and $y - z = \frac{r}{s} \in \mathbb{Q}$, then

$$\begin{aligned} x - z &= x - \left(y - \frac{r}{s}\right) \\ &= (x - y) + \frac{r}{s} \end{aligned}$$

  which is rational, since $(x - y)$ and r/s are both rational and the sum or difference of rational numbers is rational.

(f)
- It is NOT reflexive. For any x, we have $x - x = 0$, which is rational, not irrational.
- It is symmetric. If $x - y \notin \mathbb{Q}$, then $y - x = -(x - y) \notin \mathbb{Q}$.
- It is NOT transitive. Let $x = \sqrt{2} + 1$, $y = 1$ and $z = \sqrt{2} + 1 (= x)$. Then xRy and yRz, but $x - z = 0 \in mathbbQ$ so x is not related to z.

(g) The key to this problem is that this relation is the empty set; no numbers can be related to any other, because $(x - y)^2 \geq 0$ for all x and y. Hence the defining formula for xRy can never be satisfied.
- It is NOT reflexive. For any x, we have $(x - x)^2 = 0$ which is not less than zero.
- It is symmetric. The implication *If* xRy *then* yRx is true since the antecedent is always false. Put differently, the test for symmetry never fails because there's nothing to apply it to!
- It is transitive for the same reason as it is symmetric. It is impossible for xRy and yRz to be true, so the statement *If* xRy *and* yRz *then* xRz is trivially true.

If you don't like those last two explanations, I can at least say this: nobody is claiming this is a useful relation. It just gives us a chance to push the boundaries and check what the definitions really say.

(h)
- It is reflexive. For any x, we have $|x - x| = 0 \leq 2$.
- It is symmetric. If $|x - y| \leq 2$ then $|y - x| \leq 2$.
- It is NOT transitive. Let $x = 0$, $y = 2$, and $z = 4$. Check that this provides a counterexample.

6.12 Define the following relations on $A = \{a, b, c\}$. (Can you think of real world examples which satisfy any of these?)

(d) $R = \{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\} \subset A \times A$ works.

(e) $R = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\} \subset A \times A$ works.

(f) $R = \emptyset$ works.

6.20
- It is reflexive. For any x, we have $x - x = 0 = 3k$ for $k = 0$.
- It is symmetric. If $x - y = 3k$ then $y - x = 3(-k)$.

- It is transitive. If $x - y = 3k$ and $y - z = 3n$, then

$$x - z = x - (y - 3n) = (x - y) + 3n = 3k + 3n = 3(k + n)$$

  Which again fits the pattern of 3 times an integer. Note that I had to use two different integers, k and n, because it's not necessarily the case that the same k works for both $x - y$ and $y - z$.

To understand this equivalence relation, it might help to rewrite $x - y = 3k$ as $x = y + 3k$. In other words, x and y are related if they differ by a multiple of 3. This is like the clock arithmetic we discussed in class, but with a 3 hour clock instead of a 12 hour clock. There are three equivalence classes,

$$\begin{aligned} E_0 &= \{\ldots, -3, 0, 3, 6, 9, \ldots\} \\ E_1 &= \{\ldots, -2, 1, 4, 7, 10, \ldots\} \\ E_2 &= \{\ldots, -1, 2, 5, 8, 11, \ldots\} \end{aligned}$$

Notice that E_5 is the same as E_2.

6.23 (a) This is reflexive: $(a, b)R(a, b)$ because $ab = ab$ for all a and b. It is symmetric: if $ab = cd$ then $cd = ab$, so $(a, b)R(c, d)$ implies $(c, d)R(a, b)$. It is also transitive; you can check the details, but it comes down to this: if $ab = cd$ and $cd = ef$, then $ab = ef$.

(b) $E_{(9,2)} = \{(a, b) \mid (a, b)R(9, 2)\} = \{(a, b) \mid ab = 18\}$. Since a and b have to be natural numbers, this amounts to $E_{(9,2)} = \{(1, 18), (2, 9), (3, 6), (6, 3), (9, 2), (18, 1)\}$.

(c) $E_{(1,2)} = \{(a, b) \mid ab = 2\} = \{(1, 2), (2, 1)\}$ is one possible answer.

(d) $E_{(1,4)} = \{(a, b) \mid ab = 4\} = \{(1, 4), (2, 2), (4, 1)\}$ is one possible answer.

(e) $E_{(1,6)} = \{(a, b) \mid ab = 6\} = \{(1, 6), (2, 3), (3, 2), (6, 1)\}$ is one possible answer.

(f) Now $E_{(9,2)} = \{(x, y) \in \mathbb{R}^2 \mid xy = 18\}$, which is a hyperbola.

6.25 (a) This is reflexive: $(a, b)R(a, b)$ because $ab = ba$ for all a and b. It is symmetric because $ay = bx$ implies $xb = ya$. It is also transitive; check the details and ask us if you have questions.

(b) Here's the definition of an equivalence class under this relation:

$$\begin{aligned} E_{(a,b)} &= \{(x, y) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \mid ay = bx\} \\ &= \{(x, y) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \mid x = \frac{a}{b} y\} \end{aligned}$$

where we've used the fact that $b \neq 0$ when dividing. So it's the set of all points in the cartesian product whose coordinate have the ratio $1 : \frac{a}{b}$.

6.26 (a) R must include $\{(a, a), (b, b), (c, c)\}$ or it fails to be reflexive. However, this three element set is also symmetric and transitive [trivially, like 6.11(g)], so it suffices. Each number is equivalent only to itself, so S is partitioned into three equivalence classes: $\{a\}$, $\{b\}$ and $\{c\}$.

(b) R must again include $\{(a, a), (b, b), (c, c)\}$ or it fails to be reflexive. Now we need to add (a, b), which means we must add (b, a) to preserve symmetry. The relation $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ is reflexive and symmetric, and passes the transitivity test as well. It splits S into two equivalence classes: $\{a, b\}$ and $\{c\}$.

(c) R must again include $\{(a,a),(b,b),(c,c)\}$. Now we add (a,b) and (b,c). We must include (a,c) to preserve transitivity, but there's more! We have to include (b,a) and (c,b) for symmetry, but then cRb and bRa, so we need (c,a) to preserve the transitive property. (We also need it for symmetry, since (a,c) is included.) That leaves us with

$$R = \{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c),(b,a),(c,b),(c,a)\}$$

Note that R now includes all nine elements of $S \times S$, and there is just one equivalence class: $\{a,b,c\}$.