(Math 3283W : solutions to skills problems due 10/23)

11.3 (a) By (A5), $x + (-x) = 0$, and by (A2), we may reverse the order: $(-x) + x = 0$. Again by (A5), applied to $(-x)$, there exists a unique real number $-(-x)$ such that $-x + -(-x) = 0$. Since $-(-x)$ is unique with this property, and we saw above that x has this property, we conclude $-(-x) = x$.

(c) First we show that $\frac{1}{x} \neq 0$ whenever $x \neq 0$. Suppose $\frac{1}{x} = 0$. Then by (M5), $x \cdot \left(\frac{1}{x}\right) = 1$. But this means $x \cdot 0 = 1$. By Theorem 11.1(b), $x \cdot 0 = 0$. We conclude that $0 = 1$, contradicting (M4); thus $\frac{1}{x} = 0$ is impossible, forcing $\frac{1}{x} \neq 0$. We can now apply (M5) to $\frac{1}{x}$, deducing the existence of a unique real number $\frac{1}{1/x}$ such that $\frac{1}{x} \cdot \frac{1}{1/x} = 1$. As in part (a), by the uniqueness of a $\frac{1}{1/x}$ with this property, it suffices to show x has the property too. We already know, by (M5), that $x \cdot \frac{1}{x} = 1$, and (M2) allows us to reverse the order: $\frac{1}{x} \cdot x = 1$. Thus x has the defining property of $\frac{1}{1/x}$, and since $\frac{1}{1/x}$ is unique with this property, we now know $\frac{1}{1/x} = x$.

(e) Suppose $x \neq 0$. By the trichotomy axiom (O1), we have two cases: (Case 1) $x > 0$ and (Case 2) $x < 0$. In Case 1, we apply axiom (O4): since $0 < x$ and $x > 0$, we get $0 \cdot x < x \cdot x$, so $0 < x^2$ (by Theorem 11.1(b), $0 \cdot x \underset{(M2)}{=} x \cdot 0 = 0$; notice that our "x" is 0, our "y" is x, and our "z" is x in (O4)). In Case 2, we use $-x$ instead of x (by Theorem 11.1(e), if $x < 0$, then $-x > -0 \underset{(A5)}{=} 0$) and repeat our reasoning: by (O4), since $-x > 0$,

$$\underset{\underset{0}{\parallel}}{0 \cdot (-x)} < (-x)(-x) \underset{\text{Thm } 11.1(c)}{=} -(-x^2) \underset{\text{part (a)}}{=} x^2.$$

Thus $0 < x^2$ in either case, and we're done.

11.6 (a) To prove $||x|-|y|| \leq |x-y|$, it's enough (by definition of absolute value) to prove the <u>two</u> inequalities $|x|-|y| \leq |x-y|$ and $-|x-y| \leq |x|-|y|$. (In general, for $\alpha, \varepsilon \in \mathbb{R}$, "$|\alpha| \leq \varepsilon$" is equivalent to $-\varepsilon \leq \alpha \leq \varepsilon$.)

On the one hand, $|x| = |x+y-y| = |(x-y)+y| \underset{\substack{\text{triangle}\\\text{inequality}}}{\leq} |x-y|+|y|$,

so $\underline{|x|-|y| \leq |x-y|}$.

On the other, $|y| = |y+x-x| = |(y-x)+x| \underset{\text{triangle}}{\leq} |y-x|+|x|$,

so $|y|-|x| \leq |y-x| = |-(y-x)| = |x-y|$,

i.e. $\underline{|x|-|y| = -(|y|-|x|) \geq -|x-y|}$. (multiplying both sides by -1 reverses the inequality)

We've now proven both inequalities, so we're done.

(b) This is an easy consequence of (a): $|x|-|y| \leq ||x|-|y|| \underset{\text{part (a)}}{\leq} |x-y| < c$ implies $|x| < |y|+c$ by adding $|y|$ to both sides.

(c) Since $|x-y| < \varepsilon$ for all $\varepsilon > 0$, we can conclude $|x-y| \leq 0$ (use Theorem 11.7, with $|x-y|$ in place of "x" and 0 in place of "y": surely, if $|x-y| < \varepsilon$, we can say $|x-y| \leq \varepsilon$ too). But all absolute values are non-negative, so $|x-y| \geq 0$ as well, i.e. $|x-y| = 0$. This only happens if $x-y=0$, that is, if $x=y$.

11.7 We prove the claim by induction on n, the base case of $n=1$ ($|x_1| \leq |x_1|$ $\forall x_1 \in \mathbb{R}$) being obvious.

Suppose, for some $K \in \mathbb{N}$, that $|x_1 + \cdots + x_K| \leq |x_1| + \cdots + |x_K|$ for any real numbers $x_1, \ldots, x_K$.

Now let $y_1, \ldots, y_{K+1}$ be any $K+1$ real numbers.

We have
$$\begin{aligned} |y_1 + \cdots + y_{K+1}| &= |(y_1 + \cdots + y_K) + y_{K+1}| \\ &\leq |y_1 + \cdots + y_K| + |y_{K+1}| \quad \text{(by the triangle inequality for 2 numbers (Thm. 11.9(d)))} \\ &\leq |y_1| + \cdots + |y_K| + |y_{K+1}|, \quad \text{(by the inductive hypothesis for } K \text{ numbers)} \end{aligned}$$

so the claim is true for any $K+1$ real numbers.

Thus it's true for any n real numbers (any $n \in \mathbb{N}$) by induction.

11.11 (a) To avoid writing the same sentence about "same sign of the leading coefficient" repeatedly, we introduce some notation: if f is a polynomial (with real coefficients, in a single variable x), define

$$\sigma(f) = \begin{cases} 1 & \text{if the leading coefficient of } f \text{ is positive} \\ -1 & \text{'' '' '' negative} \\ 0 & \text{if } f = 0, \text{ the zero polynomial.} \end{cases}$$

Then the ordering on $\mathbb{F}$ given in Example 11.5 is: $\frac{f}{g} > 0$ if and only if $\underline{\sigma(f) = \sigma(g)}$ $(\neq 0)$, i.e. f and g's leading coefficients have the same sign.

Note the following facts: (I) The leading coefficient of $-f$ is the negative of the leading coefficient of f, so $\underline{\sigma(f) = -\sigma(-f)}$.

<u>If</u> $\sigma(f) = \sigma(g)$,

then... (II) $\sigma(hf) = \sigma(hg)$ for any h, since the leading coefficient of hf (resp. hg) is the product of the leading coefficients of h and f (resp. g);

(III) $\sigma(f+g) = \sigma(f)$ $(= \sigma(g))$: if, for instance, f <u>and</u> g have positive leading coefficients, so does $f+g$ (same for negative).

[(II) and (III) do not hold without the assumption that $\sigma(f) = \sigma(g)$.]

We now prove that "<" satisfies the order axioms (O1)-(O3).

((O1), trichotomy, in schematic form.)

Given $\frac{f}{g}$ and $\frac{p}{q}$ in $\mathbb{F}$ $\dashrightarrow$ <u>CASE 1</u> $\boxed{\frac{f}{g} = \frac{p}{q}}$ (done!)

$\dashrightarrow$ <u>CASE 2</u> $\frac{f}{g} \neq \frac{p}{q}$, so $\frac{f}{g} - \frac{p}{q} = \frac{fq - pg}{gq} \neq 0$, so $fq - pg,\ gq \neq 0$.

(Thus $\sigma(fq-pg)$, $\sigma(gq)$ are ± 1.)

<u>CASE 2A</u> $\sigma(fq - pg) = \sigma(gq)$.

Then by definition of "> 0", $\frac{f}{g} - \frac{p}{q} > 0$, so $\boxed{\frac{f}{g} > \frac{p}{q}}$

<u>CASE 2B</u> $\sigma(fq-pg) \neq \sigma(gq)$, so $\sigma(fq-pg) = -\sigma(gq)$.

By (I) above, $\sigma(-(fq-pg)) = \sigma(gq)$, so $\frac{-(fq-pg)}{gq} = \frac{p}{q} - \frac{f}{g} > 0$, i.e. $\boxed{\frac{p}{q} > \frac{f}{g}.}$

(11.11(a), cont.) ((02), transitivity of equality)

Suppose $\frac{f}{g}, \frac{p}{q}, \frac{h}{k} \in \mathbb{F}$ are such that $\frac{f}{g} < \frac{h}{k}$ and $\frac{h}{k} < \frac{p}{q}$. We'll show $\frac{f}{g} < \frac{p}{q}$.

Our hypotheses imply that $\frac{h}{k} - \frac{f}{g} = \frac{hg - fk}{kg} > 0$ and $\frac{p}{q} - \frac{h}{k} = \frac{pk - hq}{qk} > 0$. By definition of "$> 0$", this means $\sigma(hg - fk) = \sigma(kg) \neq 0$ and $\sigma(pk - hq) = \sigma(qk) \neq 0$. Now consider the sum of these two expressions:

$$\frac{p}{q} - \frac{f}{g} = \left(\frac{p}{q} - \frac{h}{k}\right) + \left(\frac{h}{k} - \frac{f}{g}\right) = \frac{pk - hq}{qk} + \frac{hg - fk}{kg}.$$

We find a common denominator for the sum on the right:

$$\frac{pk - hq}{qk} + \frac{hg - fk}{kg} = \frac{g(pk - hq)}{gqk} + \frac{q(hg - fk)}{gqk}$$

Since $\sigma(hg - fk) = \sigma(kg)$, $\sigma(q(hg - fk)) = \sigma(gqk)$ by (II).

Since $\sigma(pk - hq) = \sigma(qk)$, $\sigma(g(pk - hq)) = \sigma(gqk)$ by (II).

Thus $\sigma(q(hg - fk)) = \sigma(g(pk - hq))$, since both are equal to $\sigma(gqk)$.

By (III) we can say $\sigma(q(hg - fk) + g(pk - hq)) = \sigma(q(hg - fk)) = \sigma(g(pk - hq))$,

i.e. $\sigma(q(hg - fk) + g(pk - hq)) = \sigma(gqk)$ by the work above.

This means (by definition of "> 0") that $\frac{q(hg - fk) + g(pk - hq)}{gqk} > 0$, since its numerator and denominator have leading coefficients with the same sign.

Simplifying this fraction, we get $\frac{q(hg - fk) + g(pk - hq)}{gqk} = \frac{gpk - qfk}{gqk} = \frac{k(gp - fq)}{k(gq)}$. We haven't altered top or bottom in any way - just rewritten them - so the signs of the leading coefficients are the same.

Thus $\sigma(k(gp - fq)) = \sigma(k(gq))$. By (II) run in reverse (make sure you see why it works in reverse: think about the contrapositive of its converse), $\sigma(gp - fq) = \sigma(gq)$, so (by definition of "$>$" 0) we get

$$\frac{gp - fq}{gq} = \frac{p}{q} - \frac{f}{g} > 0, \text{ as claimed.}$$

(11.11(a), cont.) (O3) Suppose that $\frac{f}{g}, \frac{h}{k}, \frac{p}{q} \in \mathbb{F}$ and that $\frac{f}{g} < \frac{h}{k}$. We need to see that

$\frac{f}{g} + \frac{p}{q} < \frac{h}{k} + \frac{p}{q}$. But this (by definition) means $\left(\frac{h}{k} + \frac{p}{q}\right) - \left(\frac{f}{g} + \frac{p}{q}\right) > 0$, and the left-hand side is just $\frac{h}{k} - \frac{f}{g}$. We've assumed $\frac{f}{g} < \frac{h}{k}$, so $\frac{h}{k} - \frac{f}{g} > 0$ by definition, and we're done.

(No signs necessary here!)

(b, c) Here, there's no real strategy other than trial and error (or looking at the powers, in (b)):

we get

$$-x^3 < 3 - x < 5 < x + 2 < x^2$$

and

$$\frac{x+1}{x^2-2} < \frac{x+2}{x^2-1} < \frac{x^2-2}{x+1} < \frac{x^2+2}{x-1}$$

can check these 3
by direct calculation:

$\frac{f}{g} < \frac{h}{k}$ means $\frac{h}{k} - \frac{f}{g} = \frac{hg - fk}{gk}$

has same leading coefficient signs
on top & bottom.