§5: Basic Set Thy

Set Thy: Seems tedious at first, but essential. As you progress to higher level courses, the language of set thy replaces arithmetic!

Def$^{\underline{n}}$ A set is an unordered collection of objects called elements. Write $x \in A$ to denote that x is an elt (or member) of A.

If A has finitely many elt,

$$|A| = \# \text{ of elts in } A$$
$$= \text{cardinality of } A.$$

(Else A is infinite).

Ways to define sets

listing elts $A = \{1, 2, 3, 4, 5\}$

$B = \{\bigcirc, \triangle, \square\}$.

defining property

shorter: $\{x > 0\}$

$$C = \{x \mid x > 0\} = \{x : x > 0\}$$

Notes 1) A "universal set" is often implied or assumed.

A: integers? B: shapes? C: real #'s?

Standard Names

$\mathbb{N}$ = natural #'s = $\{1, 2, 3, 4, \ldots\}$

$\mathbb{Z}$ = integers = $\{\ldots -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ = ratl #'s

$\mathbb{R}$ = real #'s

$\mathbb{C}$ = Complex #'s

$\mathbb{F}_{p^n}$ = finite field w/ p^n elts.

$[-1, 3) = \{x \in \mathbb{R} \mid x \geq -1 \wedge x < 3\}$

$\emptyset = \{\}$

<u>**Subsets**</u> A is a subset of B, $A \subset B$,
if $x \in A \Rightarrow x \in B$.

<u>**Ex**</u> $B = \{1, 2, 3, 4\}$.

$A = \{1, 3, 4, 2\} \subset B$ — non-proper

$A = \{1, 3\} \subset B$ — Proper

$A = \{1, 2, 4, 5\}$ No! ($5 \notin B$)

$\emptyset \subset B$. — proper

A subset of B is PROPER if it doesn't contain <u>all</u> the elts of B.

i.e. $\subset$ but $\neq$

<u>**Notes**</u> ① to prove $A = B$, must show
$A \subset B$ <u>and</u> $B \subset A$ ($x \leq y \wedge y \leq x$

② Some books use $\subset$, $\subseteq$ for proper, proper or equal. (Think $<$, $\leq$)

<u>Most</u> use $\subset$ for both.
Our book uses $\subseteq$ for both.

Forming new sets from old.

Intersection $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Venn Diagrams:

Union $A \cup B = \{x \mid x \in A \vee x \in B\}$

⚠ $A \cap B \subseteq A \cup B$

Complement $\bar{A} = A^c = \{x \mid \sim(x \in A)\} = \{x \notin A\}$

Our book: if X is universal set

$\bar{A} = A^c = X \setminus A$.

Set Difference $A - B = A \setminus B = \{x \mid x \in A, x \notin B\}$

= complement of B in A

Ex In $\mathbb{N}$, $A =$ even #s, $B = \{1, 2, 3, \dots, 10\}$.

$A \cap B = \{2, 4, 6, 8, 10\}$

$A \cup B = \{1, 2, 3, \dots, 10\} \cup \{12, 14, 16, \dots\}$

$\bar{A} =$ odds

$A \setminus B = \{12, 14, 16, \dots\}$

$B \setminus A = \{1, 3, 5, 7, 9\}$

$A \cup \emptyset = A$

$B \cap \emptyset = \emptyset$

Ex **Prove** $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

like: $x \cdot (y + z) = (x \cdot z) + (x \cdot y)$

Pf: We must show $X \cap (Y \cup Z) \subset (X \cap Y) \cup (X \cap Z)$
and $\supset$.

Lets prove $\subset$ first. Let x be any elt in $X \cap (Y \cup Z)$. We want to show $x \in (X \cap Y) \cup (X \cap Z)$.

$x \in X \cap (Y \cup Z)$, so $x \in X$ and $Y \cup Z$.

In particular, $x \in Y \cup Z$ means $\underline{x \in Y}$ <u>OR</u> $\underline{x \in Z}$ (or both).

If $x \in Y$, then $x \in X \cap Y$ since it's in both X and Y.

Similarly, if $x \in Z$, then $x \in X \cap Z$.

Hence $x \in X \cap Y$ or $x \in X \cap Z$ (or both).

$\Rightarrow x \in (X \cap Y) \cup (X \cap Z)$.

Thus $X \cap (Y \cup Z) \subset (X \cap Y) \cup (X \cap Z)$.

You try: $\supset$

Warmup: Prove $A\setminus B = (U\setminus B)\setminus(U\setminus A)$ where U is the universal set.

Key: $A\setminus B \subseteq (U\setminus B)\setminus(U\setminus A)$, $(U\setminus B)\setminus(U\setminus A) \subseteq A\setminus B$

<u>Pf</u> (No words - generally bad!)

$x\in A\setminus B \Leftrightarrow x\in A$ and $x\notin B$.

$\Leftrightarrow x\in(U\setminus B)$ and $x\notin(U\setminus A)$

$\Leftrightarrow x\in(U\setminus B)\setminus(U\setminus A)$

Hence LHS $\subseteq$ RHS.

<u>Next</u>, let $x\in(U\setminus B)\setminus(U\setminus A)$, which means $x\in U\setminus B$ and $x\notin U\setminus A$, i.e. $x\notin B$ and $x\in A$. Hence $x\in A\setminus B$, and RHS $\subseteq$ LHS

Having shown both inclusions, we see that the sets are equal.

→ Alternatively, could change each $\Rightarrow$ in 1st half to $\Leftrightarrow$. ⚠ Rarely- not all stmts in all pfs are iff.

Indexed Sets

Often we use families of sets.

Ex $A_n = [-n, n]$, $n \in \mathbb{N}$.

$A_1 = [-1, 1]$ $A_{100} = [-100, 100]$ n: index

$A_2 = [-2, 2]$ $\mathbb{N}$ = index set

set of indices.

We'll often use notation similar to

$$\sum_{n=1}^{5} a_n = a_1 + a_2 + a_3 + a_4 + a_5$$

when dealing with indexed sets.

Ex $\bigcup_{n=1}^{5} A_n = A_1 \cup A_2 \cup \cdots \cup A_5$

$= [-1,1] \cup [-2,2] \cup \cdots \cup [-5,5]$

$= [-5, 5]$

To prove: Let $x \in [-5,5]$, show it's in $\bigcup_{n=1}^{5} A_n$ and vice versa. (two inclusions).

Let $x \in [-5,5] = A_5$. Since A_5 is a subset of $A_1 \cup A_2 \cup \cdots \cup A_5$, $x \in \bigcup_{n=1}^{5} A_n$.

Ex $\bigcap_{n=1}^{\infty} A_n = [-1,1] \cap [-2,2] \cap \cdots$
$= [-1,1]$

Pf: First let $x \in \bigcap_{n=1}^{\infty} A_n$, so $x \in A_n \ \forall n$.
In particular, $x \in A_1 = [-1,1]$.
Thus $x \in [-1,1]$ and $\bigcap_{n=1}^{\infty} A_n \subset [-1,1]$

Conversely, let $x \in [-1,1]$. Then
$x \in [-n,n]$ for all $n \in \mathbb{N}$
i.e. $x \in A_n$ for all $n \in \mathbb{N}$
$\Rightarrow x \in A_1 \cap A_2 \cap A_3 \cap \cdots = \bigcap_{n=1}^{\infty} A_n$.
Thus $[-1,1] \subset \bigcap_{n=1}^{\infty} A_n$.

Hence $\bigcap_{n=1}^{\infty} A_n = [-1,1]$.