§6. Relations

<u>Or</u>: "Sophisticated def$^{\underline{n}}$s of things you (mostly) already know."

$\exists$ five important def^{s} in section 6:

1. ordered pair
2. ★ cartesian product
3. relation
4. ★ equivalence relation
5. (partition into) equiv. classes.

<u>Sets</u> are unordered: $\{1,2\} = \{2,1\} = \{2,1,1\}$

But often order is important — with points/vectors, for example,

$$(1,2) \neq (2,1)$$

<u>Option!</u> Define a new "ordered set" where order of elts matters.

Option 2 Mathematicians like building everything out of a few basic objects. Can we define "ordered pair" w/ sets?

Def The ordered pair (a,b) is the set

$$(a,b) = \{\{a\}, \{a,b\}\} = \{\{a,b\}, \{a\}\}$$
$$= \{\{b,a\}, \{a\}\}$$

Ex $(1,2) = \{\{1\}, \{1,2\}\}$

$(2,1) = \{\{2\}, \{1,2\}\} = \{\{2\}, \{2,1\}\}$

In fact, $(a,b) \neq (b,a)$ unless $a = b$.

Thm 6.2 $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$.

Pf: $\Leftarrow$ $(a,b) = \{\{a\}, \{a,b\}\}$
$= \{\{c\}, \{c,d\}\}$ since $\begin{matrix} a=c \\ b=d \end{matrix}$
$= (c,d)$.

Def **Cartesian Product** of sets A, B is

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}.$$

i.e. all ordered pairs w/ 1st coord in A, 2nd coord in B.

Std Example Points/vectors (x,y) live in

$$\mathbb{R}^2 = \underline{\mathbb{R}} \times \underline{\mathbb{R}} = \{(x,y) \mid \underline{x \in \mathbb{R}},\ y \in \underline{\mathbb{R}}\}$$

$$\mathbb{R}^3 = \mathbb{R} \times (\mathbb{R} \times \mathbb{R}) = \{(x,(y,z)) \mid x \in \mathbb{R},\ (y,z) \in \mathbb{R}$$

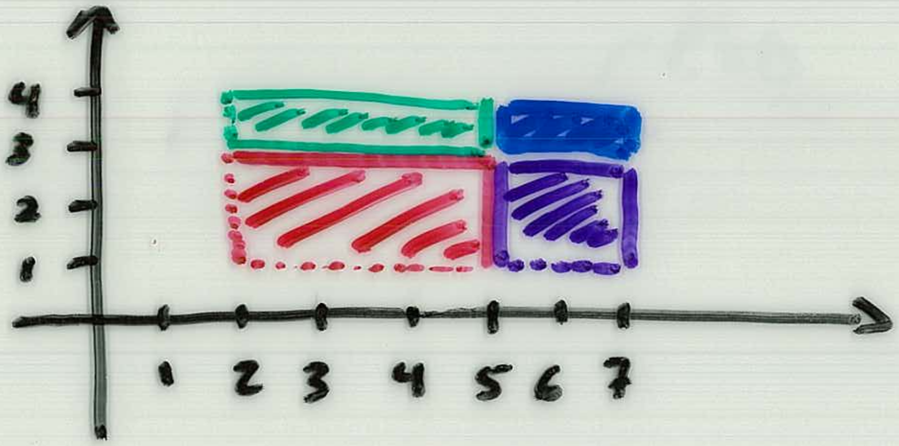
(shorthand) $= \{(x,y,z) \mid x,y,z \in \mathbb{R}\}.$

$$\{1,2\} \times \mathbb{N} = \{(n,m) \mid n \in \{1,2\},\ m \in \mathbb{N}\}$$

$$\{1,2\} \times \{\bigcirc, \triangle\} = \{(1,\bigcirc), (1,\triangle), (2,\bigcirc), (2,\triangle)\}.$$

Ex $A = (2,5]$ $B = (1,3]$

$C = [5,7]$ $D = [3,4]$



$A \times B = \{(x,y) \mid x \in (2,5], y \in (1,3]\}$

$A \times D = \{2 < x \le 5, 3 \le y \le 4\}$

$C \times B =$

$C \times D =$

Often we're interested in relationships b/w elements of sets:

numbers: $x = y$ $1 > 0$ $e \le \pi$

shapes: $\sim$ (similar triangles)

$\cong$ (congruence)

<u>Technical Def</u> A <u>relation b/w A, B</u> is a subset $R \subset A \times B$. If $(a,b) \in R$, we write aRb.

Very often $A = B$, and we say R is a rel'n on A.

<u>Ex</u> $A = B = \mathbb{R}$, $A = B = \{\text{triangles}\}$

<u>Ex</u> $P = \{A, B, C, D\}$

$S = \{\text{house, car, boat, bike}\}$

A owns house.
B owns house. } married.
C owns boat and bike.
D owns nothing.

Represented as a relation,

owns = {(A, house), (B, house), (C, boat), (C, bike)}.

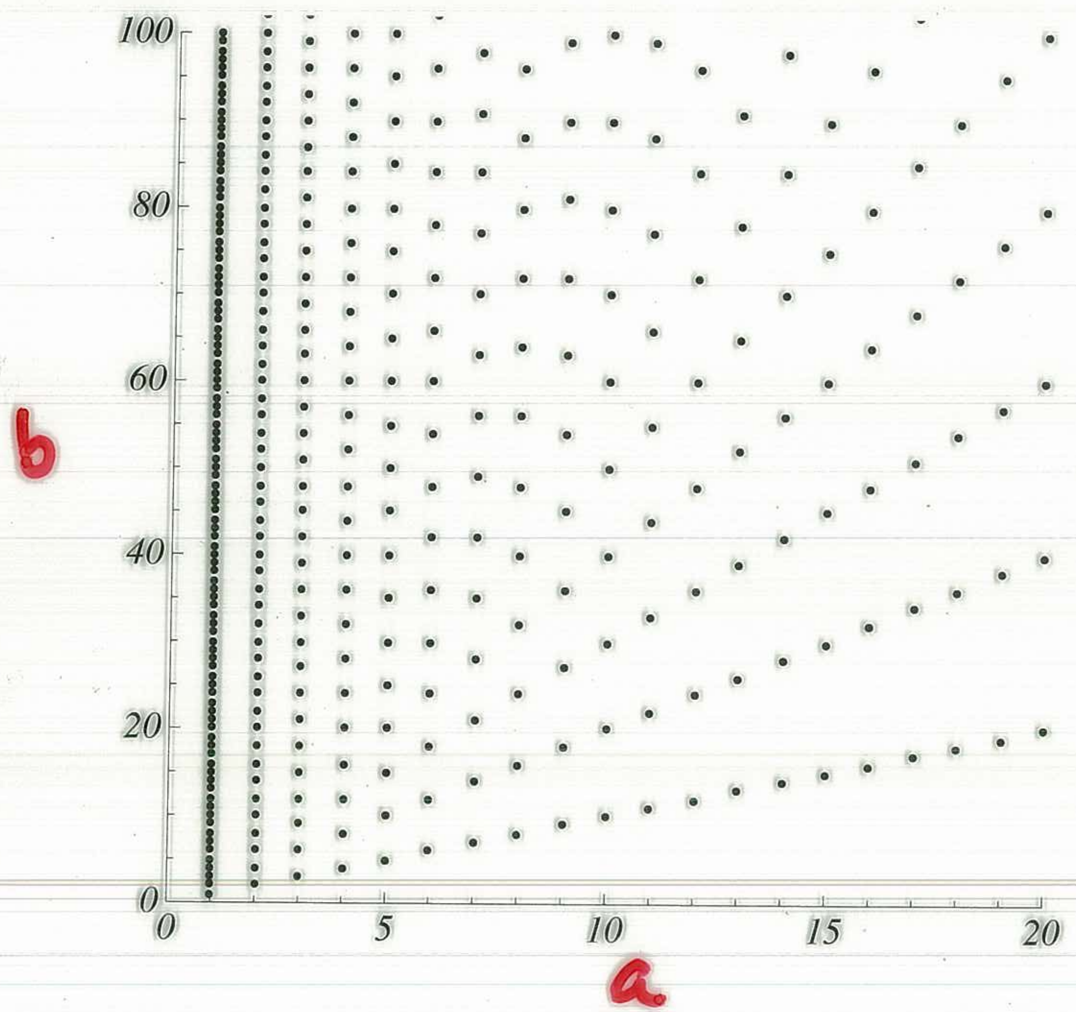
<u>Ex</u> $=$ is a rel'n on $\mathbb{Z}$, represented by following subset of $\mathbb{Z} \times \mathbb{Z}$:

$\{\ldots, (-1,-1), (0,0), (1,1), (2,2), \ldots\}$

Ex Rel'n on $\mathbb{N}$: "divides". write $|$, not R.

$a | b$ if a divides b with no remainder (a is a factor of b).

What's the subset of $\mathbb{N} \times \mathbb{N}$ for $|$?



Last time: A relation b/w A, B is a subset $R \subseteq A \times B$. Often $A = B$ and $R \subseteq A \times A = A^2$ is a rel'n on A.

Ex Clock arithmetic. Rel'n on $\mathbb{Z}$.

n related to m, $n \equiv m$ if $m = n + 12k$, some $k \in \mathbb{Z}$.

Alt, if n, m have same remainder upon division by 12.

As a subset of $\mathbb{Z} \times \mathbb{Z}$, this is

$$\{(1,1), (1,13), (1,25), \ldots\}$$
$$\cup \{(1,-11), (1,-23), \ldots\}$$
$$\cup \{\text{etc} \ldots\}$$

Def An <u>equivalence rel'n</u> R is a rel'n on a set S which satisfies these conditions $\forall x, y, z \in S$:

1. Reflexive: xRx
2. Symmetric: $xRy \Rightarrow yRx$
3. Transitive: xRy and $yRz \Rightarrow xRz$.

Ex which of the following sets, rel'ns give <u>equivalence</u> rel'ns?

- $\mathbb{N}, <$ No, not reflexive. $2 \not< 2$.
- $\mathbb{N}, \leq$
  - Is reflexive: $x \leq x \;\; \forall x \in \mathbb{N}$.
  - Not symmetric. $x = 2, y = 4$ ctrex.
- $\mathbb{N}, =$ Yes!
- $\mathbb{N}, \mid$ (divides) · No, not symmetric.

- polygons, $\cong$ Yes!
  congruent

- polygons, $\sim$ Yes!
  similar.

- lines, $\parallel$ Yes!

- lines, $\perp$ No! (symm, not reflx, not trans.)

Equivalence relations are our way of generalizing "equality" to other contexts (like geometry).

Eq. Rel'ns give us a way to group elts:

<u>Def</u> Given an eq. rel R on a set S,
The eq. class of $x \in S$ is

$$E_x = \{ y \in S \mid xRy \}$$

<u>Ex</u> Clock arithmetic: $E_8 = \{\ldots, -16, -4, 8, 20, 32, \ldots\}$

$E_{-4} = E_{20} = E_8$

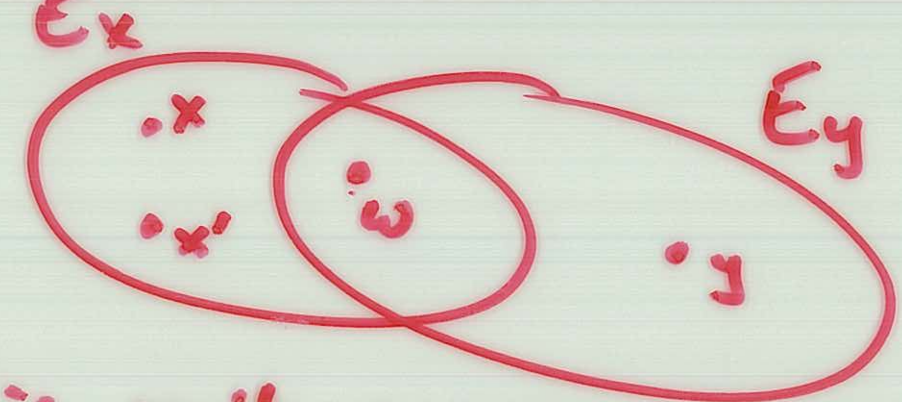
<u>Ex</u> lines, ∥. Let $\ell : y = 2x+1$.

$E_\ell = \{ y = 2x + a \mid a \in \mathbb{R} \}$

<u>Proposition</u> Different eq classes are disjoint.
(i.e. if $x, y \in S$, either $E_x = E_y$ or $E_x \cap E_y = \emptyset$.)

Pf Let $x, y \in S$.
<u>Either</u> $E_x \cap E_y = \emptyset$ (done!)
<u>or</u> $\exists\, w \in E_x \cap E_y$ (need to show $E_x = E_y$)
xRw because $w \in E_x$.
yRw b/c $w \in E_y$. (wRy by symm.)
Thus xRw, $wRy \Rightarrow xRy$.
($x \in E_y$, $y \in E_x$. You show $E_x = E_y$)

Beware the potentially mis-leading pic in book:



This <u>can't</u> happen —

$$xRw,\ wRy \Rightarrow xRy$$

$$\Rightarrow E_x = E_y.$$

Last time A **partition** of S is a collection $\mathcal{P}$ of non-empty subsets of S $\ni$:

(a) $\forall x \in S \ \exists A \in \mathcal{P} \ni x \in A$

(b) $A, B \in \mathcal{P} \Rightarrow A = B$ or $A \cap B = \phi$

Ex $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{2, 3, 5, 7\}$, $B = \{1\}$, $C = \{4, 6, 8, 10\}$

$D = \{9\}$

$\mathcal{P} = \{A, B, C, D\}$ is a part'n of S.

Ex 100 fourth graders sign up for a summer soccer league.

Assigning teams = creating a part'n.

(a) $\Rightarrow$ everybody is on a team

(b) $\Rightarrow$ no overlap, i.e. no kid is on multiple teams.

Breaking up a set into disjoint pieces has a name:

<u>Def</u> A partition of set S is a collection $\mathcal{P}$ of non empty subsets of S s.t.

(a) every $x \in S$ is in a piece A of $\mathcal{P}$, $x \in A$.

(b). $A, B \in \mathcal{P} \Rightarrow A = B$ or $A \cap B = \emptyset$.

$\mathcal{P}$ is a set of sets!

<u>Ex 6.13</u> S: UMN students

$xRy \Leftrightarrow x, y$ born in same year.

R is an equivalence rel'n. (Check)

The E_x's form a partition of S:

$\mathcal{P} = \{$ {students born in 2012},

{ —— " —————— 2011},

⋮

{ —— " —————— 2000},

⋮

$\}$

Thm 6.15 (Partition) Let R be eq rel'n on S.

(a) $\{E_x \mid x \in S\}$ is a partition of S.

(b) Let $\mathcal{P}$ be a partition of S. Then

$x R y \iff x, y \in$ same piece of $\mathcal{P}$

is an eq rel'n.

Pf: "Follow your nose." Unravel definitions.

Last Example: Recall Clock arithmetic: two integers x, y related if $x = y + 12k$, $k \in \mathbb{Z}$.

$(x - y = 12k)$

$$\mathcal{P} = \{ E_0 = \{\ldots, -12, 0, 12, 24, \ldots\},$$
$$E_1 = \{\ldots, -11, 1, 13, 25, \ldots\}$$
$$\vdots$$
$$E_{11} = \{\ldots, -1, 11, 23, 35, \ldots\}.\}$$