Recap

- $|S|$ is cardinality or card'l # of the set S.

- $|S| = |T|$ iff $S \sim T$, i.e. $\exists$ bijection $f: S \to T$. (or $T \to S$)

- $|S| \le |T|$ iff $\exists$ an <u>injection</u> $S \to T$.

  (injection might "leave out" elts in codomain.)

Ex $|I_2| \le |I_3|$ where $I_n = \{1, 2, \dots, n\}$

$f(1) = 1$

$f(2) = 2$.

$|\mathbb{N}| \le |\mathbb{Z}|$.

The identity fn $f: \mathbb{N} \to \mathbb{Z}$, $n \mapsto n$ is an injection.

⚠ We already know $|\mathbb{N}| = |\mathbb{Z}|$?!

$f: \mathbb{N} \to \mathbb{Z}$ which sort also injection — bijection, even!

$1 \mapsto 0$
$2 \mapsto 1$
$3 \mapsto -1$
$4 \mapsto 2$
$5 \mapsto -2$
$\vdots$

$f^{-1}: \mathbb{Z} \to \mathbb{N}$ also an injection $\Rightarrow |\mathbb{Z}| \le |\mathbb{N}|$.

Schröder-Bernstein Thm (Take-Home Writing problem) says $|S| \le |T|, |T| \le |S| \Rightarrow |S| = |T|$.

Sets may be:

- finite ($\sim I_n$)
- denumerable ($\sim \mathbb{N}$) countable
- uncountable. ($\mathbb{R}$) infinite.

Ex $[0,1] \sim [0,1)$.

Must find bijection $[0,1] \to [0,1)$

⚠ Avoid the "tyranny of the continuous fn." — bijection need not be continuous.

Key: identity fn $f(x) = x$ almost a bij'n $[0,1] \to [0,1)$.

Need to choose value for $f(1)$.

Set $f(1) = 1/2$.

$f(1/2) = 1/3$.

$f(1/3) = 1/4$.

$f(1/4) = 1/5$

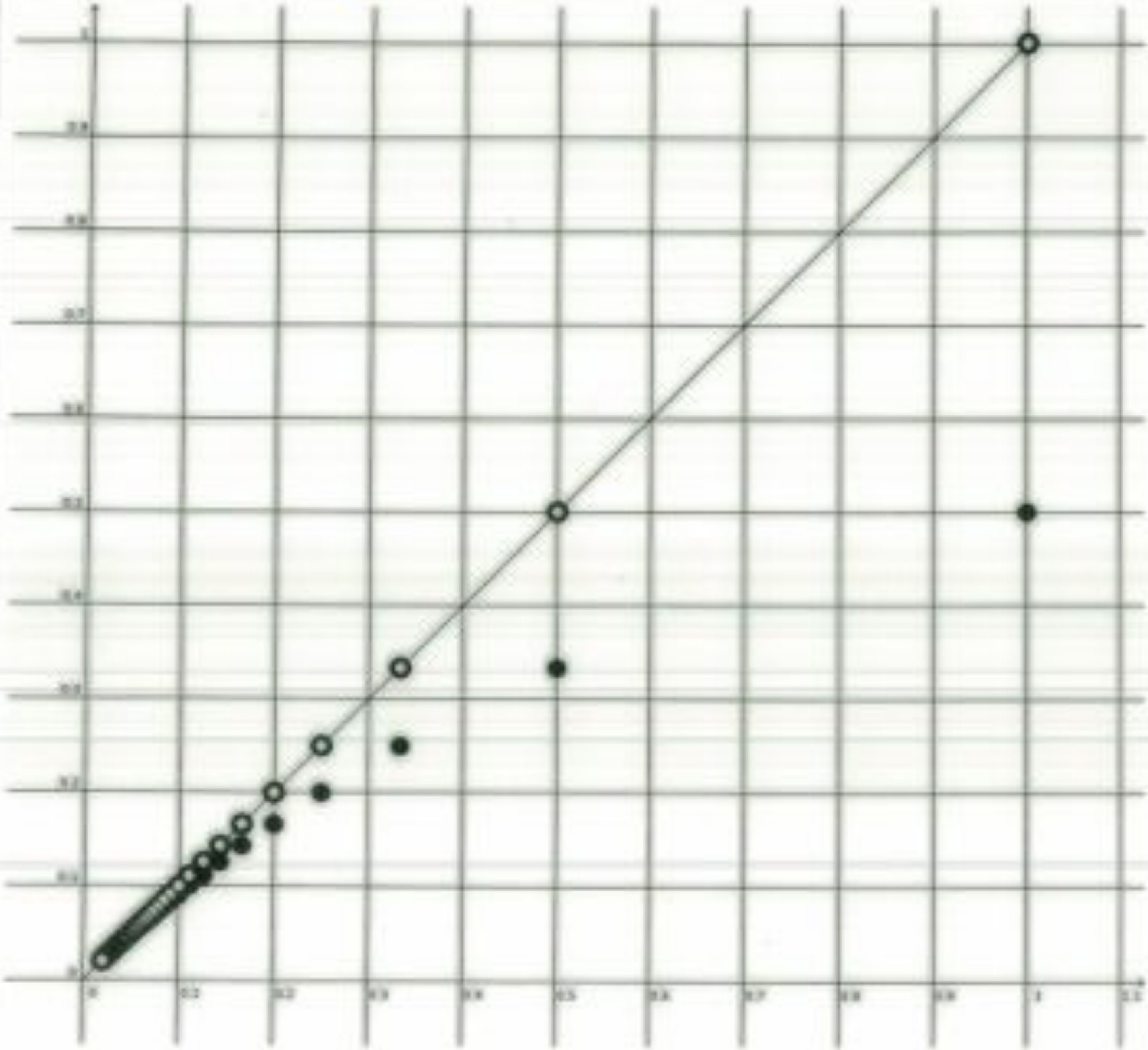
$\vdots$

Think: Hotel ∞ with room #s

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$$

Have everybody move one down

$$\underline{\text{So}}\ f(x) = \begin{cases} \frac{1}{n+1}, & x = \frac{1}{n},\ n \in \mathbb{N} \\ x, & \text{otherwise.} \end{cases}$$



Thm 8.9 S ct'ble, $T \subseteq S \Rightarrow T$ ct'ble

Pf: We don't know if S is finite or denumerable. If T is finite, we're done. So suppose T not finite. ($\Rightarrow S$ not finite either $\Rightarrow S$ denumerable).

S denumerable $\Rightarrow$ line up elts of S: $\{f(1), f(2), f(3), \ldots\}$ for $f: \mathbb{N} \to S$ bij'n. Rename $f(n)$ as s_n: $S = \{s_1, s_2, s_3, \ldots\}$

Slightly hand-wavy proof:
Define $g: \mathbb{N} \to T \subseteq S$ by

$g(1) =$ 1st elt of T which appears in list.

$g(2) =$ 2nd elt of T in

$g(3) =$ 3rd elt of T in

etc. (You check: bij'n).

Thm 8.10 The following are equiv ("TFAE")

(a) S ct'ble

(b) $\exists$ injection $f: S \to \mathbb{N}$

(c) $\exists$ surjection $f: \mathbb{N} \to S$

TFAE: start w/ any of a, b, c, prove the others are true.

6 proofs !?!

$$\begin{tikzcd} (a) \arrow[r, Leftrightarrow] \arrow[dr, Leftrightarrow] & (b) \arrow[d, Leftrightarrow] \\ & (c) \end{tikzcd}$$

Shortcut

$$\begin{tikzcd} (a) \arrow[r, Rightarrow] & (b) \arrow[d, Rightarrow] \\ & (c) \arrow[ul, Rightarrow] \end{tikzcd}$$

(3 pfs)

or

$$\begin{tikzcd} (a) \arrow[r, Leftrightarrow] \arrow[dr, Leftrightarrow] & (b) \\ & (c) \end{tikzcd}$$

(4 pfs)

Pf: (a) $\Rightarrow$ (b)

If S ct'ble, it means we can "list" the elts in order (shown in class)

S finite: $S = \{s_1, s_2, s_3, \ldots, s_n\}$

Define $f: S \to \mathbb{N}$ as $f(s_1) = 1$, $f(s_2) = 2$, $\vdots$, $f(s_n) = n$

f inj'n (check)

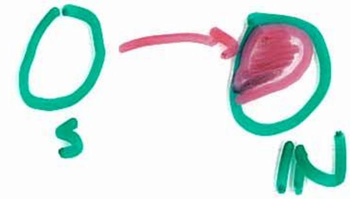
S infinite (so $S \sim \mathbb{N}$), $S = \{s_1, s_2, \dots\}$

Now define $f: S \to \mathbb{N}$ via
$f(s_1) = 1$
$f(s_2) = 2$
$f(s_3) = 3$
$\vdots$

AH: $S \sim \mathbb{N}$ means $\exists$ bijection $g: S \to \mathbb{N}$, and g is injection.

(b) $\Rightarrow$ (a) ($\exists$ injection $S \to \mathbb{N}$ $\Rightarrow$ S ct'ble)

If $f: S \to \mathbb{N}$ injective, then $f: S \to \text{rng}(S) = f(S)$ is bijection.



$\Rightarrow S \sim \underbrace{f(S) \subseteq \mathbb{N}}$ ct'ble

$\Rightarrow f(S)$ ct'ble (Thm 8.9)

$\Rightarrow S$ ct'ble

(a) $\Rightarrow$ (c) (S ct'ble $\Rightarrow$ $\exists$ surj'n $\mathbb{N} \to S$)

If S finite, $S \sim I_n$, $S = \{s_1, s_2, \dots, s_n\}$

Define $g: \mathbb{N} \to S$
$g(1) = s_1$
$g(2) = s_2$
$\vdots$
$g(n) = s_n$
$g(n+1) = s_n$
$g(n+2) = s_n$
$\vdots$
surj'n.

If S infinite, $S \sim \mathbb{N}$, $\exists$ bij'n $g: \mathbb{N} \to S$ which is surjective.

Finally,

<u>Def</u> Given a set S, its **power set** is $\mathcal{P}(S)$ = set of all subsets of S.

<u>Ex</u> $S = \{a, b\}$ $\quad \mathcal{P} = \{\emptyset, \{a\}, \{b\}, \overset{S}{\overset{\shortparallel}{\{a, b\}}}\}$

<u>Ex</u> $\mathcal{P}(\emptyset) = \{\emptyset\} \neq \emptyset$ (weird...)

$\emptyset = \{\ \}$
$\{\emptyset\} = \{\{\ \}\}$

<u>Ex</u> $S = \mathbb{N}$

$$\begin{aligned}
\mathcal{P}(\mathbb{N}) = \{\emptyset,\ &\{1\}, \{2\}, \{2\}, \ldots \\
&\{1,2\}, \{1,3\}, \{1,4\}, \ldots \\
&\{2,2\}, \{2,3\}, \ldots \\
&\vdots \\
&\{1,2,3\} \ldots \\
&\vdots \\
&\ldots, \mathbb{N}\}.
\end{aligned}$$

<u>Thm 8.18</u> $|S| < |\mathcal{P}(S)|$

<u>Ex</u> $\underset{\aleph_0}{\underset{\shortparallel}{|\mathbb{N}|}} < \underset{|\mathbb{R}|}{\underset{\shortparallel}{\underset{c}{\underset{\shortparallel}{|\mathcal{P}(\mathbb{N})|}}}} < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots$