§ 8 Cardinality

Recall the Hotel Infinity, with rooms 1, 2, 3, 4, ...

When full, we can still make room for a new arrival by moving everybody over one room.

"Room changing function" is $f(n) = n+1$.

Two new arrivals? No problem. $f(n) = n+2$.

Everybody from a different (full) Hotel ∞?

$f(n) = 2n$.

$1 \mapsto 2$
$2 \mapsto 4$
$3 \mapsto 6$
$4 \mapsto 8$

Frees up rooms 1, 3, 5, ... for ∞'ly many arrivals

Ok, so infinity is weird.

It gets worse when we compare sizes of sets. With finite sets, say

$$A = \{1, 2\}$$
$$B = \{\bigcirc, \triangle, \square\}$$

we say $|A| = 2$, $|B| = 3$ to denote # of elmts in A, B. Note. $|B| > |A|$

Which of the following sets are "larger" than the others?

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$

$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$

$\mathbb{R} \setminus \mathbb{Q}$ = irrationals.

$\mathbb{R}$

$\mathbb{C}$

Bijections are the main tool for dealing with sizes of infinite sets.

($\Rightarrow$ injections, surjections, bijections are **really** important.)

Def Two sets S, T are **equinumerous**, $S \sim T$, if $\exists$ bijection b/w them.

every $x \in T$ hit by exactly one input.

idea: if $S \sim T$, they are in 1:1 correspondence and $|S| = |T|$.

Ex $\{1,2,3\} \sim \{a,b,c\}$ Define f by:

$f(1) = a$

$f(2) = b$

$f(3) = c$

then f is a bijn $\Leftarrow$

$\{1,2,3,4\}$ and $\{a,b,c\}$ NOT equinum's.

No bijn possible:

Ex: choose $f(1), f(2), f(3)$ — no new output left for $f(4)$.

Ex $\{a,b,c,d\}$ and $\mathbb{N}$.

Not equinumerous. No bijection $\{a,b,c,d\} \to \mathbb{N}$. Could choose 4 different outputs for a,b,c,d in $\mathbb{N}$. Not surjective.

Prop $\sim$ is an equiv rel'n

Pf
- Reflexive: $id_A : A \to A$ is a bij'n, so $a \mapsto a$
  $A \sim A$. (for any set A).
- Symmetric Let $f: A \to B$ be a bij'n, so $A \sim B$. $f^{-1}: B \to A$ is bij'n, so $B \sim A$.
- Transitive If $A \sim B$, $B \sim C$, $\exists$ bij'ns $f: A \to B$
  $g: B \to C$
  $\Rightarrow g \circ f : A \to C$ is bij'n (§7)
  so $A \sim C$.

Ex $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \ldots\}$

Define $f : \mathbb{N} \to \mathbb{N}_0$
$n \mapsto n-1$

$f(1) = 0$
$f(2) = 1$
$f(3) = 2$
$f(4) = 3$
$\vdots$

Every elt in $\mathbb{N}_0$ is hit by exactly 1 input $\Rightarrow f$ bij'n

$\Rightarrow \mathbb{N} \sim \mathbb{N}_0$

Ex $\mathbb{N}, \mathbb{Z}$

$\mathbb{N} \sim \mathbb{Z}$ via this bij'n:

$f(1) = 0$
$f(2) = 1$
$f(3) = -1$
$f(4) = 2$
$f(5) = -2$
$f(6) = 3$
$f(-7) = -3$
$\vdots$

$f : \mathbb{N} \to \mathbb{Z}$
$n \to (-1)^n \lfloor \frac{n}{2} \rfloor$

"floor" or "round down" fn.

Ex $\mathbb{N}$, $\mathbb{Q}$.

Here, work with $\mathbb{Q}^+ = \{\frac{p}{q} \geq 0\}$

If $\mathbb{N} \sim \mathbb{Q}^+$, then

$\mathbb{N} \sim \mathbb{Q}$ (similar to $\mathbb{N} \sim \mathbb{Z}$)

Write non-neg rat'ls in a grid:

START

$$
\begin{array}{cccccc}
\frac{0}{1}, & \frac{1}{1}, & \frac{2}{1}, & \frac{3}{1}, & \frac{4}{1}, & \cdots \\
\frac{0}{2}, & \frac{1}{2}, & \frac{2}{2}, & \frac{3}{2}, & \frac{4}{2}, & \cdots \\
\frac{0}{3}, & \frac{1}{3}, & \frac{2}{3}, & \frac{3}{3}, & \frac{4}{3}, & \cdots \\
\vdots & & & & &
\end{array}
$$

$\frac{p}{q}$ in p^{th} col (st. at 0)
q^{th} row.

Define $f: \mathbb{N} \to \mathbb{Q}^+$:

$f(1) = 1^{\text{st}}$ # on path $= \frac{0}{1} = 0$

$f(2) = 2^{\text{nd}}$ # $= \frac{1}{1} = 1$

$f(3) = 3^{\text{rd}}$ unique # on path $= \frac{1}{2}$

$f(4) = 4^{\text{th}}$ unique # $= \frac{2}{1} = 2$

$f(5) = 3 \quad f(6) = \frac{1}{3} \quad f(7) = \frac{1}{4}, \ldots$

Def^{s} involving the cardinal # of a set, denoted $|S|$.

1. $I_n = \{1, 2, 3, \ldots, n\}$
2. We define $|I_n| = n$ and say $|S| = n$ if $S \sim I_n$. Also, $|\phi| = 0$ and $|S| = |T|$ if $S \sim T$.
3. Set S is FINITE if $S = \phi$ or $|S| = n$.
4. Set S is INFINITE if it is not finite. If $|S|$ is not finite, called transfinite
5. S is COUNTABLE if $|S| = n$ or $|S| = |\mathbb{N}| = \aleph_0$

<u>Note</u> Countable means you can put elts in order (1^{st}, 2^{nd}, ...) and count them:

$\lvert S \rvert = n$	$\lvert S \rvert = \lvert \mathbb{N} \rvert$
$\exists$ bij'n $f : \underline{I_n} \to S$	$\exists$ bij'n $g : \mathbb{N} \to S$
$1 \mapsto f(1)$	$1 \mapsto f(1)$
$2 \mapsto f(2)$	$2 \mapsto f(2)$
$3 \mapsto f(3)$	$3 \mapsto f(3)$
$\vdots$	$\vdots$
$n \mapsto f(n)$	$\vdots$
stops after n values	$\vdots$

never stops. $\longrightarrow$

Ex $\mathbb{R}$ is not countable.

⚠ You'll show $|\mathbb{R}| = |(0,1)|$ so we'll prove: $(0,1)$ not ctb'le.

Suppose $(0,1)$ is ct'ble. Write elts in a list:

dec'l expansions

$$(0,1) = \left\{ \begin{array}{l} 0.a_{11}a_{12}a_{13}a_{14}a_{15}\cdots \\ 0.a_{21}a_{22}a_{23}a_{24}a_{25}\cdots \\ 0.a_{31}a_{32}a_{33}a_{34}a_{35}\cdots \\ \vdots \end{array} \right\}$$

Define $b \in (0,1)$ which does not appear on list (giving a contradiction) by: let $b = 0.b_1 b_2 b_3 \ldots$ where $b_i \neq a_{ii}$

Say $b_i = \begin{cases} 2 & a_{ii} = 1 \\ 1 & \text{otherwise} \end{cases}$

$\boxed{\begin{array}{l} |\mathbb{Z}| = |\mathbb{N}| \\ = |\mathbb{Q}| \neq |\mathbb{R}| \end{array}}$

b is not on list $\Rightarrow$ $\mathbb{R}$ not ct'ble

Overall

we have:

Finite: $|S| = n$

Denumerable: $|S| = |\mathbb{N}|$

Uncountable: e.g. $|\mathbb{R}|$.

Continuum Hyp.: $\nexists$ no set "in between" $\mathbb{N}$, $\mathbb{R}$.

Ok, we know $|\mathbb{N}| = |\mathbb{Q}| \neq |\mathbb{R}|$, but how do we know which is <u>larger</u>?

<u>Intuitive Def$^{\underline{n}}$</u>: $|S| \leq |T|$ if $\exists$ injection $f: S \to T$.

"left over parts" make $|T| \geq |S|$

If f injective, $|S| = |\text{range} f|$