Chapter 3: $\mathbb{R}$.

§10. $\mathbb{N}$ (!?) and induction

$\mathbb{N}$ provides nice intro to properties of number systems/sets. For example:

<u>Axiom 10.1</u> $\mathbb{N}$ is <u>well-ordered</u>, meaning

$\forall \emptyset \neq S \subseteq \mathbb{N}$, $\exists$ "least" elt $m \in S \ni$
$m \leq k \;\; \forall k \in S$.

<u>Ex</u> $S = \{10, 9, 100, 99, 1000, 999, \ldots\} \subseteq \mathbb{N}$.
9 is least elt here: $9 \leq k \;\; \forall k \in S$.

<u>Aside #1</u> After choosing smallest elt, do it again for remaining #'s. Then again. And again. This lets you write all of S in "ascending order."

$S = \{9, 10, 99, 100, 999, 1000, \ldots\}$

Aside #2 Can every set be well-ordered?!

What's the least elt of $(0,1) \subseteq \mathbb{R}$?

Well Ordering "Thm" Every set can be well ordered using some relation.

Hmmm believable?

Equivalent to

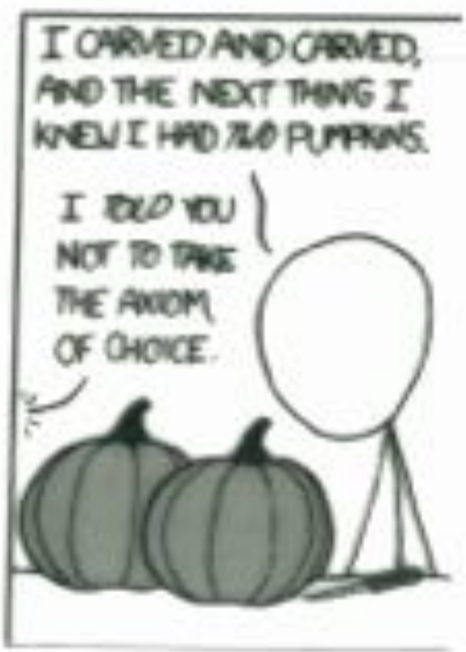
Axiom of Choice Given any infinite collection of bins (sets), we can choose one object (elt) from each.

More believable, but has weird consequences like Well ordering Thm or, ...

Banach Tarski Paradox

A sphere in $\mathbb{R}^3$ can be cut into a finite number of pieces which can be rearranged and glued back together into TWO identical copies of the original sphere. (!!!)

<u>Not</u> physically possible - pieces are "infinitely jagged" with parts that are smaller than atoms.



Thm 10.2 (Pf by Induction)

Let $P(n)$ be a stmt which is true/false for each n. If:

(a) $P(1)$ is true base, anchor

(b) $\forall k \in \mathbb{N}$, $P(k)$ true $\Rightarrow$ $P(k+1)$ true. induction step

Then $P(n)$ true for all n.

Pf Assume (a), (b) true but $\exists$ some $P(n)$ which is false.

Let $S = \{m \in \mathbb{N} : P(m) \text{ is false.}\}$

WOP of $\mathbb{N}$ says $\exists$ least elt $m \in S$ b/c by assumption $S \neq \emptyset$.

(a) $\Rightarrow$ $m \neq 1$.

m least value for which $P(m)$ false

$P(m-1)$ true, (b) $\Rightarrow$ $P(m)$ true ↯.

10.3 Obligatory Historical Example

Prove $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Ex $P(1)$: $1 = \frac{1(1+1)}{2} = \frac{2}{2} = 2.$

$P(2)$: $1+2=3$, $\frac{2(2+1)}{2} = \frac{6}{2} = 3.$

$$1 + 2 + 3 + \cdots + 999 + 1000$$

$$\underline{1000 + 999 + 998 + \cdots + 2 + 1}$$

$$1001 + 1001 + 1001 + \cdots 1001 + 1001$$

Sum is $\dfrac{1000 \cdot (1001)}{2}$

<u>Inductive Pf:</u> $P(n): 1+2+\cdots+n = \dfrac{n(n+1)}{2}$.

base: $P(1)$: already checked!

Inductive Step: Assume $P(k)$, show $P(k+1)$ is true:

$$1+2+3+\cdots+k = \frac{k(k+1)}{2}.$$

Then:

$$\underbrace{1+2+3+\cdots+k}_{=\frac{k(k+1)}{2} \text{ by } P(k)} + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k^2+k+2k+2}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Avoid:

Assume $P(k)$, then:

$$1+2+\cdots+k+k+1 = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)}{2}+k+1 = \frac{(k+1)(k+2)}{2}$$

$$\vdots$$

$$LHS = RHS$$

$a=b$

$a^2=b^2$

$a^2-b^2=b^2-b^2$

$(a-b)(a+b)=0$

$a+b=0$

$a=-b$

$a=-a$

$1=-1.$

avoid "two sided" equality proofs – too easy to divide/mult by zero, etc.