§11 Ordered Fields

$\mathbb{R}$ is a "complete ordered field," meaning:

complete: to be described later, but means nothing is "missing". e.g.:

- In $\mathbb{N}$, set of #'s b/w 0, 1 is ϕ; in $\mathbb{R}$, $(0,1)$ is uncountable!
- In $\mathbb{Q}$ we have numbers $x \in \mathbb{Q}$ s.t. $x^2 < 2$, but $\nexists\, x \in \mathbb{Q}$ s.t. $x^2 = 2$ ($\sqrt{2}$ is a real #.).

ordered: any two $x, y \in \mathbb{R}$, $x \neq y$ can be ordered, $x < y$ or $y < x$.

field we can do all arithmetic operations: $+, -, \times, \div$ (except by 0), and get another real #.

Examples

$\mathbb{N}$: we can add $\forall x,y \in \mathbb{N},\ x+y \in \mathbb{N}.$

but not subtract $10-2 \in \mathbb{N},\ 2-10 \notin \mathbb{N}.$

can multiply $\forall x,y \in \mathbb{N},\ xy \in \mathbb{N}.$

not divide $\frac{4}{2} \in \mathbb{N},\ \frac{1}{2} \notin \mathbb{N}.$

$\mathbb{Z}$: we can add, subtract, multiply two integers and get an integer. But

$\forall p,q \in \mathbb{Z}\ (q \neq 0),\ \frac{p}{q}$ not nec. in $\mathbb{Z}$.

$\mathbb{Q}$: field (like $\mathbb{R}$): can add, subtract, mult. and divide fractions and always get another fraction.

$\mathbb{R}$:= "completion" of $\mathbb{Q}$.

$\mathbb{R}$ satisfies following axioms:

field

A1. For all $x, y \in \mathbb{R}$, $x+y \in \mathbb{R}$ and if $x = w$ and $y = z$, then $x + y = w + z$.

A2. For all $x, y \in \mathbb{R}$, $x+y = y+x$. – commutativity of +.

A3. For all $x, y, z \in \mathbb{R}$, $x+(y+z) = (x+y)+z$. – associativity of +.

A4. There is a unique real number 0 such that $x + 0 = x$, for all $x \in \mathbb{R}$.

A5. For each $x \in \mathbb{R}$ there is a unique real number $-x$ such that $x + (-x) = 0$.

M1. For all $x, y \in \mathbb{R}$, $x \cdot y \in \mathbb{R}$, and if $x = w$ and $y = z$, then $x \cdot y = w \cdot z$.

M2. For all $x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$.

M3. For all $x, y, z \in \mathbb{R}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

M4. There is a unique real number 1 such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.

M5. For each $x \in \mathbb{R}$ with $x \neq 0$, there is a unique real number $1/x$ such that $x \cdot (1/x) = 1$. We also write x^{-1} in place of $1/x$.

DL. For all $x, y, z \in \mathbb{R}$, $x \cdot (y+z) = x \cdot y + x \cdot z$.

ordered

O1. For all $x, y \in \mathbb{R}$, exactly one of the relations $x = y$, $x > y$, or $x < y$ holds (trichotomy law).

O2. For all $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.

O3. For all $x, y, z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$.

O4. For all $x, y, z \in \mathbb{R}$, if $x < y$ and $z > 0$, then $xz < yz$.

How could these fail? See earlier examples (and abstract algebra).

All rules of arithmetic/algebra follow from?

Thm 11.1 Let $x, y, z \in \mathbb{R}$:

(a) $x + z = y + z \Rightarrow x = y$.

(b) $x \cdot 0 = 0$

(c) $(-1) \cdot x = -x$

(d) $xy = 0 \iff x = 0$ or $y = 0$

(e) $x < y \iff -y < -x$

(f) $x < y$, $z < 0 \Rightarrow xz > yz$

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DL. For all $x, y, z \in \mathbb{R}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

O1. For all $x, y \in \mathbb{R}$, exactly one of the relations $x = y$, $x > y$, or $x < y$ holds (trichotomy law).

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O3. For all $x, y, z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$.

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Proof of (b), $x \cdot 0 = 0$.

$x \cdot 0 = x(0+0)$ (A4)

$x \cdot 0 = x \cdot 0 + x \cdot 0$ (DL)

$x \cdot 0 + 0 = x \cdot 0 + x \cdot 0$ (A4)

$0 + \underline{x \cdot 0} = x \cdot 0 + \underline{x \cdot 0}$ (A2 – commutativity)

$0 = x \cdot 0$ (part (a) of thm)

Thm 11.7 Let $x, y \in \mathbb{R}$. If $x \le y + \varepsilon \ \forall \varepsilon > 0$, then $x \le y$.

Remark Think "limits" – $y + \varepsilon \ \forall \varepsilon > 0$ is a fancy way of writing

$$\lim_{h \to 0^+} y + h$$

In Calc I language,

$$\lim_{h \to 0} x \le \lim_{h \to 0} y + h$$

$$\Rightarrow \quad x \le y.$$

Def $|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

Read 11.9, esp (d): $|x+y| \le |x| + |y|$

$$-|x| \le x \le |x|$$
$$-|y| \le y \le |y|$$

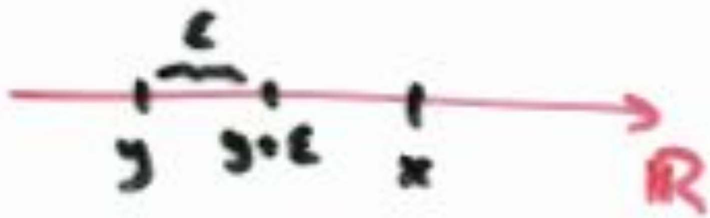
$$-(|x|+|y|) \le x + y \le |x| + |y|$$

Prove: Let $x, y \in \mathbb{R}$. If $x \le y + \varepsilon \ \forall \varepsilon > 0$ then $x \le y$.

Pf: CP: $x > y \Rightarrow \exists \varepsilon > 0$ s.t. $y + \varepsilon < x$.

Suppose $x > y$

Let $\varepsilon = \frac{x-y}{2} > 0$.



Then
$$y < y + \varepsilon = y + \frac{x-y}{2} = \frac{x+y}{2}$$
$$< \frac{x+x}{2} \text{ (b/c } x > y\text{)}$$
$$= \frac{2x}{2} = x.$$

Overall
$$y < \underbrace{y + \varepsilon < x}_{\text{as desired.}}$$

A totally different ordered field

$$\mathbb{F} = \left\{ \frac{p(x)}{q(x)} \;\middle|\; p, q \text{ polynomials}, q(x) \neq 0 \right\}$$

$$= \left\{ \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_k x^k + \cdots + b_1 x + b_0} \;\middle|\; \begin{array}{l} a_i, b_i \in \mathbb{R}, \\ b_k \neq 0 \end{array} \right\}$$

= set of rat'l fns.

Examples $\frac{1}{1}$, $\frac{0}{1}$, $\frac{0}{x}$, $\frac{0}{x^2+1}$

$$\frac{x(x-1)}{x^2+4} = \frac{x^2-x}{x^2+4}, \quad \frac{2}{3x+1}$$

Not $\frac{\sin(x)}{x^2}$, $\frac{4x^2+3x+2}{0}$.

To make $\mathbb{F}$ an <u>ordered</u> field, need to define when $\frac{p}{q} < \frac{r}{s}$. Start with:

<u>Def</u> $\dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + \cdots + a_0}{b_k x^k + \cdots + b_0}$ is "positive,"

written $\dfrac{p}{q} > 0$ iff a_n, b_k have same sign.

⚠ <u>Not</u> our normal $<$!!

<u>ONE</u> way to write this algebraically, not the only way:

$$a_n b_k > 0.$$

these are <u>NOT</u> normal $>, \le$

these are std $>, \le$ for #'s.

<u>Ex</u>

$\dfrac{x+1}{3x^2-4} > 0$ b/c $(1)(3) > 0$

$\dfrac{0}{x-1} \le 0$ b/c $(0)(1) \le 0$

$\dfrac{\pi x}{-ex^3} \le 0$ b/c $(\pi)(-e) < 0$.

Def $\frac{p(x)}{q(x)} > \frac{f(x)}{g(x)}$ if $\frac{p}{q} - \frac{f}{g} > 0$

(else $\frac{p}{q} \leq \frac{f}{g}$).

Ex Determine "sign" and order of $\frac{x}{x+2}, \frac{x}{x+1}$.

$\frac{x}{x+2} > 0$ b/c $1 \cdot 1 > 0$

$\frac{x}{x+1} > 0$ b/c $1 \cdot 1 > 0$

$$\frac{x}{x+1} - \frac{x}{x+2} = \frac{x(x+2) - x(x+1)}{(x+1)(x+2)}$$

$$= \frac{x^2 + 2x - x^2 - x}{(x+1)(x+2)}$$

$$= \frac{x}{x^2 + 3x + 2} > 0$$

$$\Rightarrow \frac{x}{x+1} > \frac{x}{x+2}$$

Question: