§12 The Completeness Axiom

Or: "How does $\mathbb{R}$ differ from $\mathbb{Q}$?"

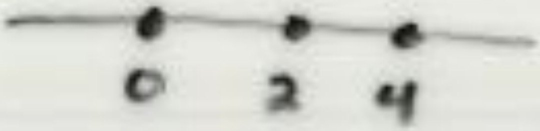
Major ideas:

1. Upper, lower bounds
★ 2. sup, inf
   - def$^{\underline{n}}$
   - finding sup S, inf S
   - proving your answers are correct
3. Completeness
4. Density of $\mathbb{Q}$.
★ 5. Archimidean Property.

We begin with bounds. Some subsets of $\mathbb{R}$ have minimum and maximum elts:

- $m \in S$ is $\min S$ if $m \le s \ \forall s \in S$
- $M \in S$ is $\max S$ if $M \ge s \ \forall s \in S$.

Ex $S = \{0, 2, 4\}$

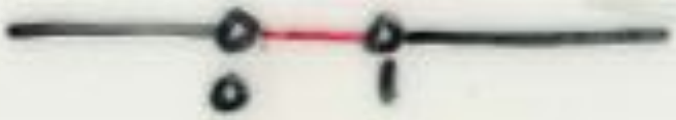


$\min S = 0$
$\max S = 4$

$S = [0, \infty)$

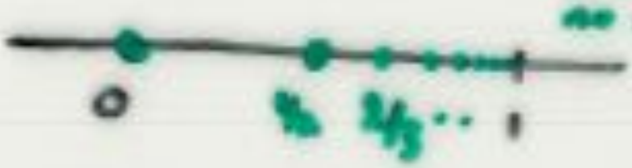
$\min S = 0$
no max.

$S = (0, 1)$



no min, no max.

$S = \{\underbrace{1 - \tfrac{1}{n}}_{= \frac{n-1}{n}} \mid n \in \mathbb{N}\}$



$\min S = 0$
no max.

More generally we have:

- m upper bound for $S \subseteq \mathbb{R}$ if
  $m \geq s \ \forall s \in S$
- m lower bound for $S \subseteq \mathbb{R}$ if
  $m \leq s \ \forall s \in S$.

<u>Ex</u> $S = \{0, 2, 4\}$

upper bds: $\underline{4}, 6, 2\pi, 7, 8, 9, \ldots$
lower bds: $\underline{0}, -e, -5, \ldots$

$S = (0,1)$ upper bds: $\underline{1}, 2, 3, 4, \ldots$
lower bds: $\underline{0}, -1, -2, \ldots$

$S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$ upper bds: $\underline{1}, 2, \ldots$
$= \{0, 1/2, 2/3, 3/4, \ldots\}$ lower bds: $\underline{0}, -1, \ldots$

∞'ly many bds, but often have "best" choice.

Observations:

① If m is an upper bound for S, so is any # larger than m.

② m lower bd $\Rightarrow$ so is any # $< m$.

③ If an upper bd $m \in S$, then $m = \max S$
—"— lower —"— ——, then $m = \min S$

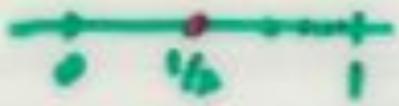
Def Let $S \subseteq \mathbb{R}$ be bounded above. (It has an upper bound - hence ∞'ly many)
The least upper bound is called the supremum of S:

$$\sup S = \operatorname{lub} S.$$

if bdd below, the greatest lower bound is the infimum.

$$\inf S = \operatorname{glb} S$$

<u>Ex</u> $S=\{1-\frac{1}{n} \mid n\in\mathbb{N}\} = \{0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\dots\}$



$\inf S = \text{glb } S = 0$
$\sup S = \text{lub } S = 1.$

$S=[0,1]$ same: $\inf S = 0$
$\sup S = 1.$

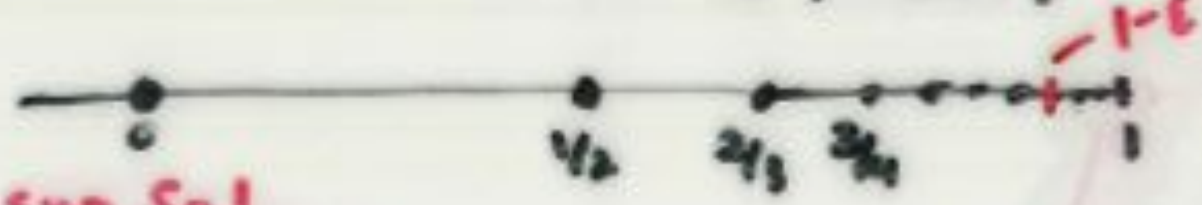
$S=(0,\infty)$ $\inf S = 0$
$\sup S$ d.n.e. (unbounded above).

So if $m=\sup S$,

(a) $m \geq s \ \forall s\in S$. (upper bd)

(b) $m'<m$ can't be an upper bd: $\exists s'\in S$ s.t. $s'>m'$

<u>Example of (b)</u> w/ $\{1-\overset{\frac{(n-1)}{n}}{\overset{=}{}}\!\!1/n \mid n\in\mathbb{N}\}$



<u>sup S=1</u>

(a) $1-\frac{1}{n} = \frac{n-1}{n} < 1$, so 1 is upper bd.

(b) harder to show: take (small) $\varepsilon>0$ s.t. $1-\varepsilon<1$ (but close) <u>show</u> it is not upper bound

<u>$\mathbb{R}$ satisfies</u>:

Completeness Axiom: Every $\emptyset \neq S \subseteq \mathbb{R}$ which is bounded above has a least upper bound.

Notes ① $\mathbb{Q}$ not complete.

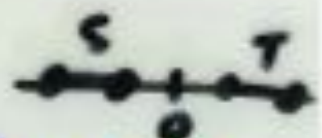
$$S = \{x \in \mathbb{Q} \mid x^2 < 2\}$$

$0 \in S$, $1 \in S$, etc $\Rightarrow S \neq \emptyset$. bdd above by 1.5, π, 10, 17.7, and so on. BUT no least upper bound — $\sqrt{2} \notin \mathbb{Q}$.

② Why not "Every $\emptyset \neq S \subseteq \mathbb{R}$ bdd below has greatest lower bd"?

Suppose $\emptyset \neq S \subseteq \mathbb{R}$ bdd below.

Let $T = \{-s \mid s \in S\}$.



T bdd above $\Rightarrow$ completeness says $\exists\ m = \sup T$

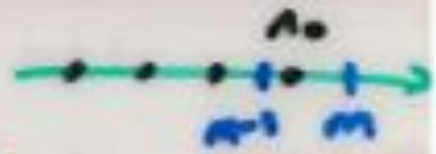
(You check: $-m = \inf S$.)

Thm 12.9 Archimidean Property of $\mathbb{R}$:

$\mathbb{N}$ has no upper bound in $\mathbb{R}$.

Pf by contradiction. Assume $\mathbb{N}$ does have an upper bd in $\mathbb{R}$. By completeness, $\exists\ m = \sup \mathbb{N}$, $m \in \mathbb{R}$.

Since m = least upper bd of $\mathbb{N}$, $m-1$ is not upper bd $\Rightarrow \exists\ n_0 \in \mathbb{N}$, $n_0 > m-1 \Rightarrow n_0+1 > m$, $n_0+1 \in \mathbb{N}$ ↯



Thm 12.10 TFAE

(*) Archimidean Property.

* (a) $\forall z \in \mathbb{R}\ \exists\ n \ni n > z$.

(b) $\forall x > 0$, $\forall y \in \mathbb{R}\ \exists\ n \in \mathbb{N} \ni nx > y$.

* (c) $\forall x > 0\ \exists\ n \in \mathbb{N} \ni 0 < \frac{1}{n} < x$.

Pf (*) $\Rightarrow$ (a): CP: $\exists\ z \in \mathbb{R} \ni z \geq n\ \forall n \in \mathbb{N}$.
$\Rightarrow \mathbb{N}$ has upper bd.

(a) $\Rightarrow$ (b) $z = y/x$

(b) $\Rightarrow$ (c) Let $y = 1$ in (b).

S10 Exam Problem $A = \left\{ \frac{n-1}{n+1} \,\middle|\, n \in \mathbb{N} \right\}$

So $A = \left\{ \frac{0}{2}, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \dots \right\}$

(i) Show A bdd above, below.

$n \in \mathbb{N} \Rightarrow n-1,\ n+1$ both $\geq 0 \Rightarrow \frac{n-1}{n+1} \geq 0$.

Also, $\forall\, n \in \mathbb{N}$, $n-1 < n+1$

$\Rightarrow \frac{n-1}{n+1} < 1$.

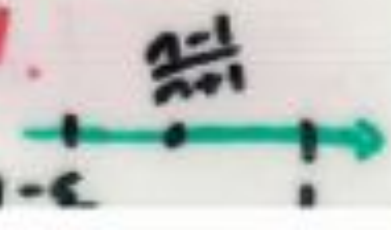
(ii) Find $l = \inf A$, $m = \sup A$.

$l = \inf A = 0$, $m = \sup A = 1$.

To prove $\sup A = 1$, must show any # less than 1 cannot be an upper bound.

Take some small $\varepsilon > 0$, so $\underline{1-\varepsilon} < 1$

Show $\exists\, n \ni 1-\varepsilon < \frac{n-1}{n+1} < 1$.



$A = \left\{ \frac{n-1}{n+1} : n \in \mathbb{N} \right\} = \left\{ \frac{0}{2}, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \dots \right\}$ 5'

Repeat to show $\sup A = 1$, show

$\forall\, \varepsilon > 0$ (small!), $\exists\, n$ s.t.

$$1 - \varepsilon < \frac{n-1}{n+1} \quad (\le 1)$$

which implies $1-\varepsilon$ not upper bd.

Use algebra to "solve" for n in terms of ε:

$$1 - \varepsilon < \frac{n-1}{n+1} = \frac{(n+1)-2}{(n+1)} = 1 - \frac{2}{n+1}$$

$$-\varepsilon < -\frac{2}{n+1}$$

"Think" Portion of this problem.

$$\varepsilon > \frac{2}{n+1} \qquad (n+1,\ \varepsilon > 0)$$

$$n+1 > \frac{2}{\varepsilon}$$

$$\Rightarrow n > \frac{2}{\varepsilon} - 1.$$

So any $\frac{n-1}{n+1}$ (with $n > \frac{2}{\varepsilon} - 1$) will be between $1-\varepsilon$ and 1 $\Rightarrow$ $1-\varepsilon$ not an upper bound.

⚠ To write out the proof, we should really use what we found with algebra and write it out in <u>reverse</u>!

Proof that $\sup \underbrace{\left\{\frac{n-1}{n+1} : n \in \mathbb{N}\right\}}_{A} = 1$

<u>Pf</u>: First note that, $\forall n \in \mathbb{N}$, $n-1 < n+1$,

"Proof" <u>Pattern</u>.

so $\frac{n-1}{n+1} < 1 \Rightarrow 1$ is upper bd of A

To show 1 is $\sup A$, we show that $\forall \varepsilon > 0$, $1-\varepsilon$ is not an upper bound. Given ε, choose

$n > \frac{2}{\varepsilon} - 1$ (by Arch. Prop)

Then

$$n+1 > \frac{2}{\varepsilon}$$
$$\varepsilon > \frac{2}{n+1}$$
$$-\varepsilon < -\frac{2}{n+1}$$
$$1-\varepsilon < 1 - \frac{2}{n+1} = \frac{n-1}{n+1} \in A.$$

$\boxed{\Rightarrow 1-\varepsilon \text{ is not an upper bound of } A.}$

Finally,

Thm 12.13 $\mathbb{Q}$ is dense in $\mathbb{R}$: $\forall x, y \in \mathbb{R}$, $x < y$, $\exists\, r \in \mathbb{Q}$ s.t. $x < r < y$.

Pf: see book.

Constructive Method: Write out decimal expansions of x, y:

3.1415692...
3.1412763...

1^{st} time they differ, "split the difference" and write as a fraction.

$$3.1413 = \frac{31{,}413}{10{,}000}.$$

Thm 12.14 $x, y \in \mathbb{R}$, $x < y \Rightarrow \exists$ irrational # ω such that $x < \omega < y$.

Pf: Thm 12.13 $\Rightarrow \exists\, r \neq 0$, $r \in \mathbb{Q}$ s.t. $\dfrac{x}{\sqrt{2}} < r < \dfrac{y}{\sqrt{2}}$

$\Rightarrow x < \sqrt{2}\, r < y$ and $\sqrt{2}\, r \notin \mathbb{Q}$.

Odds/Ends - make sure to read §12,
ask questions / see examples
in discussion.

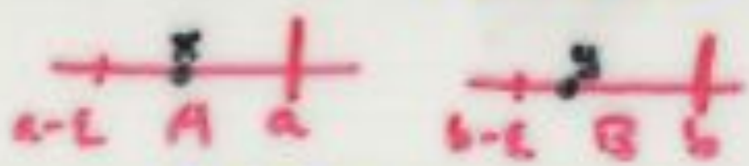
Thm Let $C = \{x+y \mid x \in A, y \in B\}$. If $\sup A$, $\sup B$ exist, then $\sup C = \sup A + \sup B$.

Pf: Let $a = \sup A$, $b = \sup B$. First show $a+b$ is an upper bd for C.

Let $z \in C$, so $z = x+y$, some $x \in A$, $y \in B$.
But $x \le a$, $y \le b \Rightarrow z \le a+b$.

Now show $a+b$ is least upper bd.
Let $c = \sup C$ (exists by completeness).
Then $c \le a+b$. We want to show $c \ge a+b$ (so $c = a+b$).

Choose $\varepsilon > 0$. $\exists\, x \in (a-\varepsilon, a]$, $y \in (b-\varepsilon, b]$



$a - \varepsilon < x$
$b - \varepsilon < y$

$a+b-2\varepsilon < x+y \le c$

$a+b < c+2\varepsilon$ $\forall \varepsilon > 0$

$\Rightarrow a+b \le c$