§12 Reminder

Def $m = \sup S$ $(S \subseteq \mathbb{R})$ iff

(a) $m \geq s \ \forall s \in S$

(b) $m' < m \Rightarrow \exists\, s^{\in S}$ s.t. $m' < s$.

⚠ In practice, we prove (b) by letting $\varepsilon > 0$ (presumably small), setting $m' = m - \varepsilon$.

Today: §13 Topology $\neq$ Topography

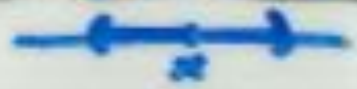
Many topics/def^{n}s here:

- ☆ 1. neighborhoods
- ☆ 2. Interior, boundary pts.
- ☆ 3. open, closed sets in $\mathbb{R}$
- 4. accumulation pts
- 5. closure of a set.

§13. Topology of $\mathbb{R}$

In this context, "topology" refers to "open and closed sets."

With sequences, limits, we talk about "points close to x." In this section we start to give that idea a careful def:



Def Let $x \in \mathbb{R}$, $\varepsilon > 0$. Then

$B_\varepsilon(x) =$ etc. $\quad N(x;\varepsilon) = \{y \in \mathbb{R} \mid |x-y| < \varepsilon\} = (x-\varepsilon, x+\varepsilon)$

is a neighborhood (nbhd) or ε-nbhd of x, with radius ε.

$N^*(x;\varepsilon) = \{y \mid 0 < |x-y| < \varepsilon\} = (x-\varepsilon, x) \cup (x, x+\varepsilon)$

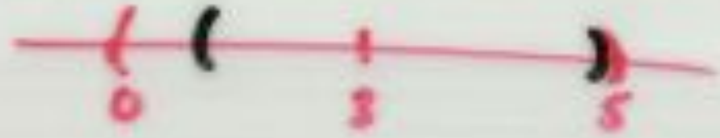
is deleted nbhd of x.

Ex $N(2;1) = (1,3)$.

$N^*(2;1) = (1,2) \cup (2,3)$

<u>Def</u> $x \in S \subseteq \mathbb{R}$ is an <u>interior</u> <u>point</u> of S if $\exists$ nbhd N of x in S: $N \subseteq S$.

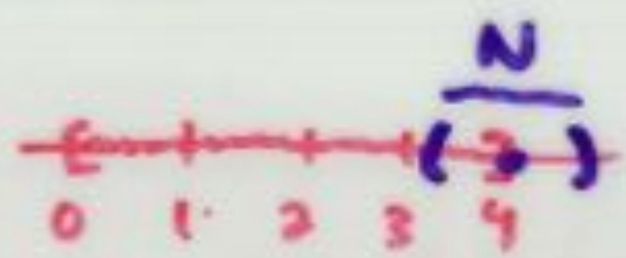
<u>Ex</u> $(0,5)$



$N(3;1) = (2,4) \subset (0,5)$ ✓
$N(3;2) = (1,5) \subset (0,5)$ ✓
$N(3;4) = (-1,7) \not\subset (0,5)$ ✗

If every nbhd of x contains pts in S $(N \cap S \neq \emptyset)$ and also contains pts <u>not</u> in S $(N \cap (\mathbb{R} \setminus S) \neq \emptyset)$, then x is a <u>boundary</u> <u>point</u> of S.

<u>Ex</u> $[0,4]$



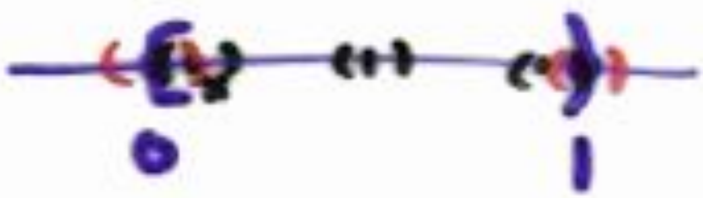
$\forall \varepsilon > 0$, $N(x;\varepsilon)$ will contain pts in $[0,4]$ and pts >4 (hence in $\mathbb{R} \setminus [0,4]$)

Ex $S = \{0, 2, 4\}$

No int. pts!
Bdy pts: 0, 2, 4.

$\forall \varepsilon > 0$, $N(2;\varepsilon) \not\subseteq S \Rightarrow 2$ not interior pt.
$N(2;\varepsilon)$ contains #'s not in S,
does contain $2 \in S$.
$\Rightarrow 2$ bdy pt.

Ex $T = [0, 1)$



Any $N(0;\varepsilon)$ will contain pts in, out of $[0,1) \Rightarrow 0$ is bdy pt.

Same for 1
(bdy pts need not be in the set.)

Choose $x \in \mathbb{R}$, $0 < x < 1$.

Compute $|x - 0|$, $|x - 1|$

Choose ε to be the smaller of the two

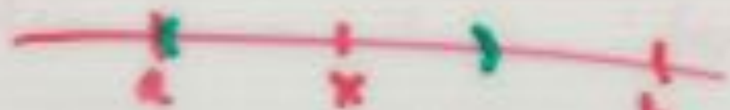
$\Rightarrow N(x;\varepsilon) \subset [0,1)$

Def $S \subseteq \mathbb{R}$ is <u>open</u> if every pt in S is an interior pt, i.e:

$$\forall x \in S \ \exists \ \varepsilon > 0 \ \ni N(x;\varepsilon) \subseteq S$$

Examples

① Any interval $(a,b) = \{a < x < b\}$ is open.



Let $x \in (a,b)$, choose

$\varepsilon = \min\{|x-a|, |x-b|\}$

Then $N(x;\varepsilon) \subseteq (a,b)$

② Since $N(x;\varepsilon) = (x-\varepsilon, x+\varepsilon)$ neighborhoods are <u>open</u>.

Can Refer to "open Nbhds".

③ $\mathbb{R}$ is open.

Let $x \in \mathbb{R}$. For <u>any</u> $\varepsilon > 0$, $N(x;\varepsilon) \subseteq \mathbb{R}$. Done.

④ $\emptyset \subseteq \mathbb{R}$ is open.

If $\underbrace{x \in S}_{\text{always FALSE}}$, then x is interior pt.

always FALSE for $S = \emptyset$, so implication is TRUE(

⑤ $S = (0,1) \cup (4,5)$

$x \in S \Rightarrow x \in (0,1)$ or $x \in (4,5)$.

If $x \in (0,1)$, $(0,1)$ is an open set, so $\exists\, \varepsilon$ s.t. $N(x;\varepsilon) \subseteq (0,1) \subseteq S$

Similar for $x \in (4,5) \Rightarrow S$ <u>open</u>.

⑥ $S = (1,6) \cap (2,9)$

$x \in S \Rightarrow x \in (1,6)$ <u>and</u> $x \in (2,9)$, both of which are open.

$\exists\, \varepsilon_1, \varepsilon_2 > 0 \ni N(x;\varepsilon_1) \subseteq (1,6)$, $N(x;\varepsilon_2) \subseteq (2,9)$. smallest is in the $\cap$'n

More generally,

<u>Thm 13.10</u>

(a) any union of open sets is open.

(b) $\cap$ of finitely many open sets is open.

<u>Pf</u>: (a) Suppose A_j is open for all $j \in J$. ($J = \mathbb{N}$? $\{1,2\}$? etc

Let $x \in \bigcup_{j \in J} A_j \Rightarrow x \in A_n$, some n. A_n open by assumption, so $\exists\, \varepsilon > 0$ s.t. $N(x;\varepsilon) \subseteq A_n$. Since $A_n \subset \bigcup_{j \in J} A_j$, we also have $N(x;\varepsilon) \subseteq \bigcup_{j \in J} A_j$.

__Pf__ (b): Suppose $A_1, \dots, A_n$ are open.

Let $x \in A_1 \cap \dots \cap A_n \Rightarrow x \in A_i$

$\Rightarrow \exists\ \varepsilon_1 > 0, \varepsilon_2 > 0, \dots, \varepsilon_n > 0$

s.t. $N(x;\varepsilon_1) \subseteq A_1$

$N(x;\varepsilon_2) \subseteq A_2$

$\vdots$

$N(x;\varepsilon_n) \subseteq A_n$

If $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$

then $N(x;\varepsilon) \subseteq A_1 \cap \dots \cap A_n$.

$\Rightarrow A_1 \cap \dots \cap A_n$ open.

⚠ Finitely many necessary here b/c $\min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots\}$ might not exist.

__Ex__ $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ __not__ an open set.


STORE
NOT CLOSED
STORE
NOT CLOSED
STORE
NOT CLOSED
STORE
?
NOT CLOSED
STORE
STORE
NOT OPEN EITHER

Def $S \subseteq \mathbb{R}$ is closed if $\mathbb{R} \setminus S = S^c$ open.

⚠ closed, open NOT opposites:

"not open" $\neq$ closed!

Think: $[0,1)$

Examples

① $[0,1] = S$. $\mathbb{R} \setminus S = S^c = (-\infty, 0) \cup (1, \infty)$ open, so S is closed.

② $(1,2]$. $\mathbb{R} \setminus (1,2] = (-\infty, \underline{1}] \cup (2, \infty)$

$(-\infty, 1]$ ↑ not open; $(2, \infty)$ ↑ open

$(1,2]$ not open, closed

1 is not interior pt of $(-\infty, 1]$

⚠ $(-\infty, 2] \cup [0, \infty)$ $\overset{\shortparallel}{\mathbb{R}}$ open

③ $\mathbb{R}$ $\mathbb{R} \setminus \mathbb{R} = \emptyset$

$\emptyset$ is open $\Rightarrow$ $\mathbb{R}$ is closed. (and open)

$\mathbb{R}$ is "clopen".

Different Characterization:

Thm $S \subseteq \mathbb{R}$ closed if it contains all its bdy points.

⚠ That's the def^{n} in book. 5 Our def^{n} then given as a Thm.

Interiors and boundaries clearly important, so we name these sets:

Def int S = set of interior pts
bd S = set of bdy pts.

Above def^{n}'s, thms can be written:

S open iff $S = \text{int } S$.
S closed iff $\text{bd } S \subseteq S$.

Cool Stuff

$\cap$'n of finitely many open sets is open, but $\cap$'n of infinitely many can be closed!

Ex $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$

$\bigcap_{n=1}^{\infty} A_n = \{0\}$ which is not an open set, b/c any $N(0;\varepsilon)$ will include pts not in the intersection.

$\{0\}^c = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ open $\Rightarrow \{0\}$ closed.

Similarly, $\cup$ of finitely many closed sets is closed, but $\cup$ of infinitely many can be open!

Ex $\bigcup_{n \in \mathbb{N}} [-n, n] = \mathbb{R}$ open (and clopen)

$\bigcup_{n \in \mathbb{N}} \left[\frac{1}{n}, 2 - \frac{1}{n}\right] = \cdots = (0, 2)$ open
not closed.

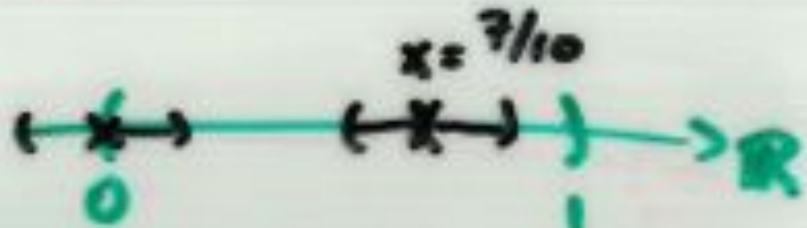
$\exists$ hybrid of interior, bdy pts:

<u>Def</u> $x \in \mathbb{R}$ is an accumulation pt of S if $\forall \varepsilon > 0$, $N^*(x;\varepsilon) \cap S \neq \emptyset$.

⚠ x need not be in S to [be] an acc. pt of S.

$\quad = (x-\varepsilon, x) \cup (x, x+\varepsilon)$

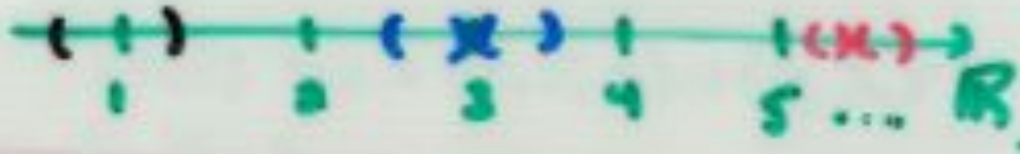
<u>Ex</u> $S = (0,1)$



$x = 7/10$ is acc pt of S.
0 is acc pt. 1 too.

Set of acc pts: $S' = [0,1]$

$\mathbb{N} \subseteq \mathbb{R}$



Any $n \in \mathbb{N}$ is bdy pt of $\mathbb{N}$ ($N(n;\varepsilon)$ will include n <u>and</u> #'s not in $\mathbb{N}$).

<u>But</u> no $n \in \mathbb{N}$ is an acc. pt.
(Set of acc pts is $\emptyset$).

$n \in \mathbb{N}$ is an isolated pt

Def closure of S = $\operatorname{cl} S = S \cup S'$

Ex $\operatorname{cl}(0,1) = (0,1) \cup [0,1] = [0,1]$

Thm Let $S \subseteq \mathbb{R}$.

(a) S closed $\Leftrightarrow S' \subset S$.

(b) $\operatorname{cl} S$ is closed.

(c) S closed iff $S = \operatorname{cl} S$.

(d) $\operatorname{cl} S = S \cup \operatorname{bd} S$.

⚠ Although

$$\underline{S \cup S'} \overset{\text{def}}{=} \operatorname{cl} S \overset{(d)}{=} \underline{S \cup \operatorname{bd} S}$$

$$\not\Rightarrow S' = \operatorname{bd} S$$

Think: $[0,1]$, where

$S' = [0,1]$

$\operatorname{bd} S = \{0,1\}$