This was all a special case of

Thm 16.8 Suppose $a_n \to 0$. If for some $k > 0$, $m \in \mathbb{N}$ we have

$$n > m \Rightarrow |s_n - s| \le k|a_n|$$

then $s_n \to s$.

Notes ① $\left|\frac{n^2+2n}{n^3-5} - 0\right| \le 4\left(\frac{1}{n}\right)$

s_n s $n>2$ (so $m=2$) $k=4$ a_n.

② This is the Squeeze Thm!

Suppose $0 \le f(x) \le h(x)$

and $\lim\limits_{x\to\infty} 0 \le \lim\limits_{x\to\infty} f(x) \le \lim\limits_{x\to\infty} h(x) =$

$\Rightarrow \lim\limits_{x\to\infty} f(x) = 0.$ (here $|s_n - s| \to$

Pf Given $\varepsilon > 0$, $a_n \to 0 \Rightarrow \exists\, N_1$ s.t. $|a_n| < \frac{\varepsilon}{k}$

Let $N = \max\{m, N_1\}$. Then $n > N$

$\Rightarrow |s_n - s| \underset{n>m}{\le} k|a_n| \underset{n>N_1}{<} k\left(\frac{\varepsilon}{k}\right) = \varepsilon.$ ∎

Q: How to prove a seq. <u>diverges</u>?

<u>Recall</u>:

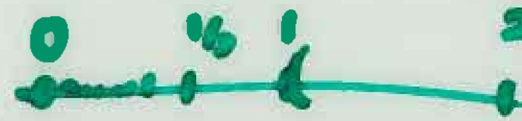
"implied $\forall$"

<u>Def</u> $s_n \to s$ iff $\forall \varepsilon > 0\ \exists N$ s.t. $n > N \Rightarrow |s_n - s| < \varepsilon$.

$s_n \not\to s$ iff $\exists\, \varepsilon > 0$ s.t. $\forall N$,
$\exists\, n > N$ s.t. $|s_n - s| \geq \varepsilon$.

i.e. $\exists\ \varepsilon > 0$ s.t. you can't ever guarantee s_n is within ε of s!

<u>Ex</u> $\frac{1}{n} \not\to 2$.



Because if $\varepsilon = 1$, then
$|\frac{1}{n} - 2| \geq \varepsilon \quad \forall\, n \in \mathbb{N}$!

Algebra: $\frac{1}{n} \leq 1$, hence ≥ 1 unit away from 2.

To show (s_n) <u>diverges</u>, must show
$s_n \not\to s \quad \forall\, s \in \mathbb{R}$.

So, to prove s_n diverges, you must exhibit some $\varepsilon > 0$ s.t., no matter how large n gets, you can't guarantee $|s_n - s| < \varepsilon$ for any s.

Ex Prove $s_n = (-1)^n$ diverges.

$(s_n) = (-1, 1, -1, 1, \ldots)$ oscillates

Suppose $s_n \to s \in \mathbb{R}$, set $\varepsilon = 1$.
(Why 1? half the dist. b/w $1, -1$.
Could use $1/2$, $1/3$, $1/10$ too...)

~~No matter how large n is,~~

$\exists$ N s.t. $n > N$ implies $|s_n - s| < \varepsilon = 1$

No matter how large n is, it still changes b/w even, odd #'s:

n odd: $|-1-s| < 1 \Rightarrow -2 < s < 0$

n even: $|1-s| < 1 \Rightarrow 0 < s < 2$.

s can't possibly satisfy both ineq's ↯.

Hence our assumption was wrong, and s_n diverges.

Thm 16.14 If $s_n \to s$, $s_n \to t$, then $s = t$.

Pf: We want to show $s = t$, $\Leftrightarrow$

$$|s-t| = 0 \Leftrightarrow |s-t| < \varepsilon, \ \forall \varepsilon > 0.$$

(Thm 11.7)

Given $\varepsilon > 0$, $\exists$ M, N s.t.

$n > N$, $|s_n - s| < \varepsilon$ or $\varepsilon/2$ ①

$n > M$, $|s_n - t| < \varepsilon$. $\varepsilon/2$

Let $K = \max\{N, M\}$, so $n > K \Rightarrow$ both

For $n > K$ we therefore have

$$\begin{aligned} |s-t| &= |(s_n - t) - (s_n - s)| && \text{(Add 0)} \\ &\le |s_n - t| + |s_n - s| && (\triangle \text{ ineq}) \\ &< \varepsilon + \varepsilon = 2\varepsilon. && ① \ \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Overall, $|s-t| < 2\varepsilon$ $\forall \varepsilon > 0$

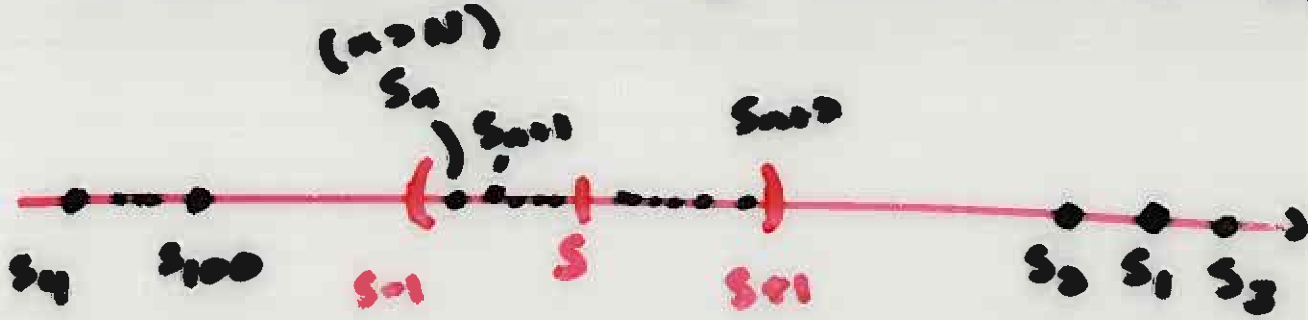
$$\Rightarrow |s-t| = 0.$$

Thm 11.7 $x \le y + \varepsilon$ $\forall \varepsilon > 0 \Rightarrow x \le y$.

$x = |s-t|$, $y = 0$ gives above claim.

Thm 16.13 A convergent seq. is bdd

Pf Suppose $s_n \to s$. For $\varepsilon = 1$ (why 1? nice round # > 0 - could use $2, \pi, \ldots$)
$\exists N$ s.t. $\forall n > N$, $|s_n - s| < \varepsilon = 1$.



Let s_N be last # outside of $N(s;1)$.
Then from s_{N+1} on, everything is within one unit of s.

Let $M = \max\{\underbrace{|s_1|, |s_2|, \ldots, |s_N|, |s|+1}_{\text{max exists b/c this set is finite.}}\}$

Then M is as large as any $|s_n|$

$\Rightarrow |s_n| < M \;\; \forall n \Rightarrow (s_n)$ bdd.