§16. Sequences and Convergence

Informally, a sequence is a list of #'s: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Formally A sequence in $\mathbb{R}$ is a function $f: \mathbb{N} \to \mathbb{R}$, where $f(i)$ is the i^{th} # in the list. So above

$f(n) = \frac{1}{n}$ $\quad f(1) = 1$, $f(2) = 1/2$ $\quad f(3) = 1/3$ etc.

Usually we avoid the fn notation and use subscripts: $a_1 = f(1)$, $a_2 = f(2)$ etc.

So for above seq., write:

$$(a_n) = (1/n)$$

$$(a_n) = (1, 1/2, 1/3, 1/4, \ldots)$$

⚠ (a_n) = the sequence $(a_1, a_2, a_3, \ldots)$
$\{a_n\}$ = the set of #'s in the seq.

Ex For $a_n = \sin(\frac{\pi}{2}n)$,
$(a_n) = (1, 0, -1, 0, 1, 0, -1, 0, \ldots)$
$\{a_n\} = \{-1, 0, 1\}$

⚠ Many, many books use $\{a_n\}$ for what our book calls (a_n).

Ways to Define/Express a Sequence

① Give a formula for n^{th} term.

$a_n = \frac{1}{n}$ gives $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$

② Give 1^{st} term(s) and recursive formula

$a_1 = 0$, $a_n = 2a_{n-1}$: $(a_n) = (0, 0, 0, 0, \ldots)$

$a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$: $(a_n) = (1, 1, 2, 3, 5, \ldots)$

③ List enough terms to establish a pattern

$(a_n) = (1, 4, 9, 16, 25, 36, \ldots)$ (sq. #'s).

⚠ Risky - what if somebody doesn't spot the pattern? Or sees a different one?

$(b_n) = (0, 7, 26, 63, \ldots)$ $b_5 = 124$

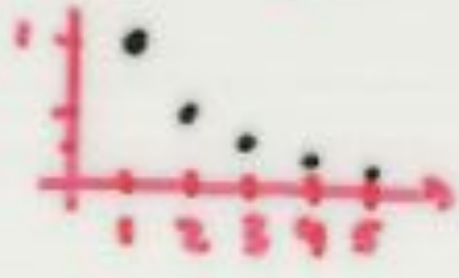
$(c_n) = (1, 11, 21, 1211, 111221, 312211, \ldots$

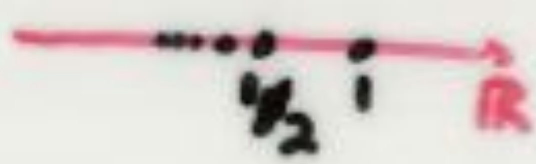
OEIS. $c_7 = 13112221$.

④ Graphically (Helpful, not rigorous)

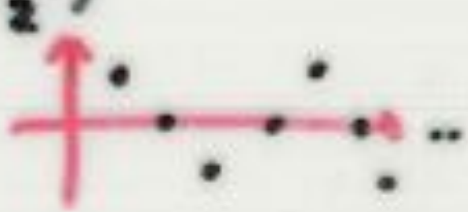
Either $\mathbb{N} \times \mathbb{R}$ or just $\mathbb{R}$

$a_n = \frac{1}{n}$





$b_n = \sin(\frac{\pi n}{2})$



Def (s_n) **converges** to $s \in \mathbb{R}$, written

$$\lim_{n\to\infty} s_n = s \quad \text{or} \quad s_n \longrightarrow s$$

or $\mathbb{R}$

if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |s_n - s| < \varepsilon$. (s_n) **diverges** if it does not converge to any $s \in \mathbb{R}$.

In words, $s_n \to s$ if

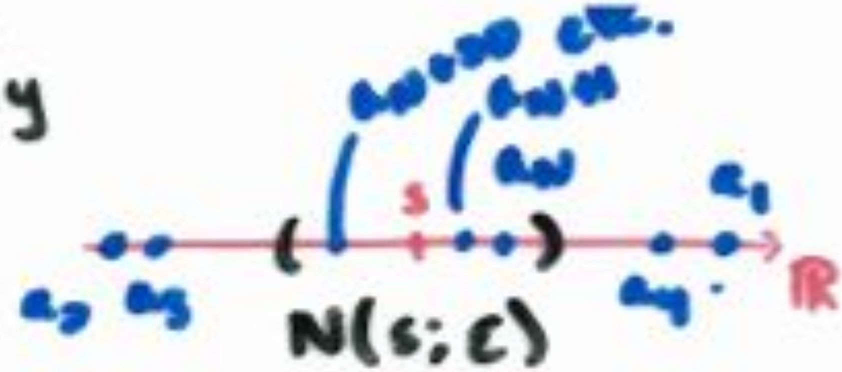
eventually every s_n is close to s.

after N^{th} term, some N.

within ε, some $\varepsilon > 0$: $|s_n - s| < \varepsilon$

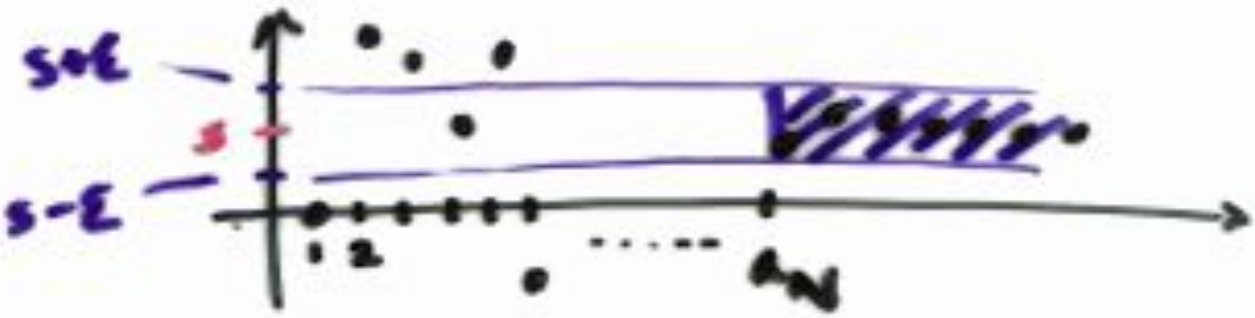
⚠ Order matters! You don't choose ε. Given ε, you must find $N \in \mathbb{N}$ (which probably depends on ε) to make the def$^{\text{n}}$ work.

Visually



for $n>N$, $a_n \in N(\text{blah})$

OR



In general, to prove $s_n \to s$,

Step 1 "Preparation" or "Thinking"

Do algebra to figure out how large n has to be to ensure $|s_n - s| < \varepsilon$. Set N equal to this value.

Step 2 "Proof": Do algebra in reverse.

Given $\varepsilon > 0$, choose $N =$ (magic #, depends on ε). Then $\forall n > N$ we have
$$|s_n - s| = \cdots < \varepsilon \quad \blacksquare.$$

Ex $(s_n) = \left(\frac{n-1}{n}\right) = \left(\frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right)$

We suspect $\lim_{n\to\infty} s_n = 1$ i.e. $s_n \to 1$.

Preparation:

$$|s_n - 1| < \varepsilon \iff -\varepsilon < s_n - 1 < \varepsilon$$

$$-\varepsilon < \frac{n-1}{n} - 1 < \varepsilon$$

$$-\varepsilon < \frac{n-1-n}{n} < \varepsilon$$

$$-\varepsilon < -\frac{1}{n} < \varepsilon$$

$$\frac{1}{n} < \varepsilon \text{ so } n > \frac{1}{\varepsilon}.$$

Proof:

For any $\varepsilon > 0$, set $N = 1/\varepsilon$.
Then $n > N \Rightarrow n > 1/\varepsilon$ and $\varepsilon > \frac{1}{n}$,
so

$$|s_n - 1| = \cdots = \left|-\frac{1}{n}\right| = \frac{1}{n} < \varepsilon. \blacksquare$$

<u>Ex</u> 16.6 Show $s_n = \frac{n^2+2n}{n^3-5} \to 0$.

<u>Algebra</u> Need to find n s.t.

$$\left|\frac{n^2+2n}{n^3-5} - 0\right| < \varepsilon$$

Tricks of the trade....

① We only care about what <u>eventually</u> happens, so we can assume $\boxed{n \geq 2}$ $\Rightarrow$ $n^2+2n > 0$, $n^3-5 > 0$

Then $\left|\frac{n^2+2n}{n^3-5}\right| = \frac{n^2+2n}{n^3-5}$.

② Try to find a simpler (larger) sequence which bounds s_n.

<u>Key</u>: $\frac{n^2+2n}{n^3-5} < \frac{p(n)}{q(n)}$ <u>if</u> $p(n) > n^2+2n$, $q(n) < n^3-5$.

Helps if $p(n)$, $q(n)$ are simple: $k \cdot n^{\text{power}}$

Find N s.t. $n > N \Rightarrow \left| \frac{n^2 + 2n}{n^3 - 5} - 0 \right| < \varepsilon$

- For large n, (specifically $\boxed{n > 2}$)

  $n^2 > 2n$ so $\underline{n^2 + 2n} < n^2 + n^2 = \underline{2n^2}$

- For large n (specifically $\boxed{n > 2}$)

  $n^3 - 5 > n^3$? never!

  $\underbrace{n^3 - 5 > \frac{1}{2} n^3}$

  Solve:

  $\frac{1}{2} n^3 > 5$

  $n^3 > 10$

  $n > \sqrt[3]{10} \approx 2.15\ldots$

n	$n^3 - 5$	$\frac{n^3}{2}$
1	-4	$1/2$
2	3	4
3	22	$27/2$
4	59	32
5	120	$125/2$

Hence for large n (i.e. $n>2$)

$$\frac{n^2+2n}{n^3-5} < \frac{2n^2}{\frac{1}{2}n^3} = 4\left(\frac{1}{n}\right)$$

(3) An N for that larger, simpler sequence will work for our messy one, too.

Want: $\left|4\left(\frac{1}{n}\right)-0\right| = \frac{4}{n} < \varepsilon.$

$\iff n > 4/\varepsilon.$

So if $n>\frac{4}{\varepsilon}$ $\boxed{\text{AND } n>2}$ *

$$\frac{n^2+2n}{n^3-5} < \frac{4}{n} < \varepsilon.$$

$n>2$ $\qquad$ $n > 4/\varepsilon.$

<u>Proof</u> that $\frac{n^2+2n}{n^3-5} \to 0$ (whew!)

Given $\varepsilon > 0$, set $N = \max\{2, 4/\varepsilon\}$.
Then $n > N \Rightarrow$

$$|s_n - 0| = \left|\frac{n^2+2n}{n^3-5} - 0\right| = \frac{n^2+2n}{n^3-5} < \frac{4}{n} < \varepsilon.$$

b/c $n > 2$ b/c $n > 2$ $n > 4/\varepsilon$

⚠ (1) We can always find n larger than $\max\{2, 4/\varepsilon\}$ by Archimedean Property (or related thm)

(2) Don't "lose the forest for the trees" - there were lots of steps here.