§17. Limit Theorems

A section full of tools!

Goal: show that these thms - many of which you know - all follow from ε, N def$^{\underline{n}}$.

Thm 17.1 Suppose $s_n \to s$, $t_n \to t$. Then

(a) $\lim (s_n + t_n) = s + t = \lim s_n + \lim t_n$

(b) $\lim (k s_n) = ks$ and $k \lim s_n$.
$\lim (k + s_n) = k + s$ $\forall k \in \mathbb{R}$.

(c) $\lim (s_n t_n) = st = (\lim s_n)(\lim t_n)$

(d) $\lim (s_n / t_n) = s/t$, provided $t \neq 0$, $t_n \neq 0$ $\forall n$.

These are "baby" versions of limit laws in calc I:

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

provided both limits exist.

Pf of (a), $s_n \to s$ and $t_n \to t \Rightarrow s_n + t_n \to s+t$

Let $\varepsilon > 0$. We need to show we can make $|(s_n + t_n) - (s+t)| < \varepsilon$.

Using Δ ineq.,

$$|(s_n - s) + (t_n - t)| \le |s_n - s| + |t_n - t|$$

Because $s_n \to s$, $t_n \to t$, we can find

$N_1 \ni |s_n - s| < \varepsilon/2$ for $n > N_1$

$N_2 \ni |t_n - t| < \varepsilon/2$ for $n > N_2$

If $n > \max\{N_1, N_2\}$ $(= N)$, then

$$\underline{|(s_n - s) + (t_n - t)|} \le |s_n - s| + |t_n - t|$$
$$\underline{<} \; \varepsilon/2 + \varepsilon/2$$
$$= \underline{\varepsilon}.$$

$\Rightarrow (s_n + t_n) \to (s+t)$.

(b)–(d): see book.

Makes life much easier...

<u>Ex</u> Prove $\frac{n^2+2n}{n^3-5} \to 0$

$$\lim\left(\frac{n^2+2n}{n^3-5}\right) \overset{\text{algebra.}}{=} \lim\left(\frac{1/n + 2/n^2}{1 - 5/n^3}\right)$$

$$\overset{17.1(d)}{=} \frac{\lim\left(1/n + 2/n^2\right)}{\lim\left(1 - 5/n^3\right)}$$

$$\overset{17.1(a)(b)}{=} \frac{\lim 1/n + 2\lim 1/n^2}{\lim 1 - 5\lim 1/n^3}$$

$$= \frac{0 + 2\cdot 0}{1 - 5\cdot 0} = \frac{0}{1} = 0.$$

Ex Prove $\lim \dfrac{4n^2-3}{5n^2-2n} = \dfrac{4}{5}$

$$\lim \frac{4n^2-3}{5n^2-2n} \neq \frac{\lim 4n^2-3}{\lim 5n^2-2n}$$

need limits on top, bottom to exist.

(Algebra) $=$

$$\lim \frac{4 - 3/n^2}{5 - 2/n}$$

17.1(d) $=$

17.1(a)

$$\frac{\lim 4 - 3/n^2}{\lim 5 - 2/n} = \frac{\lim 4 - \lim \frac{3}{n^2}}{\lim 5 - \lim 2/n}$$

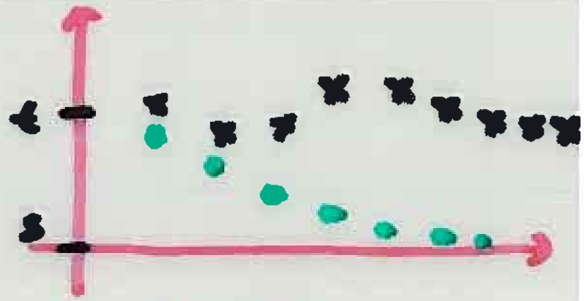
17.1(b)

$$= \frac{\lim 4 - 3 \lim \frac{1}{n^2}}{\lim 5 - 2 \lim \frac{1}{n}}$$

$$= \frac{4 - 3\cdot 0}{5 - 2\cdot 0} = \frac{4}{5}.$$

Thm 17.4 (Another "Squeeze" thm)

Suppose $s_n \to s$, $t_n \to t$. If $s_n \le t_n$ $\forall n$, then $\underline{s \le t}$.



$\bullet = s_n$

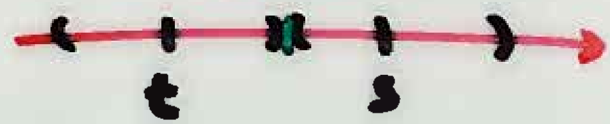
$\times = t_n$

Pf: Cool proof by contradiction.

Suppose not, that $t < s$.

Choose $\varepsilon = \dfrac{s-t}{2} > 0$

$t + \varepsilon = s - \varepsilon$



For large enough N, $(N \gg 0)$

$n > N \Rightarrow$

$$s - \varepsilon < s_n < s + \varepsilon \quad (|s_n - s| < \varepsilon)$$

$$t - \varepsilon < t_n < t + \varepsilon \quad (|t_n - t| < \varepsilon)$$

$$\Rightarrow \quad t_n < t + \varepsilon = s - \varepsilon < s_n$$

i.e. $t_n < s_n$. ↯

Corollary: $t_n \ge s_n \ge 0$, $t_n \to 0 \Rightarrow s_n \to 0$.

What if, instead of $\frac{1}{n}, \frac{1}{n^2}$, etc. we have seq's with factorials: $n! = n(n-1)\cdots 3\cdot 2$

or expon'l fns: $e^x, 2^x, \ldots$

useful tool:

<u>Thm 17.7</u> Suppose $s_n \geq 0$, ratios $\left(\frac{s_{n+1}}{s_n}\right)$ converge to L. If $L < 1$ then $s_n \to 0$.

<u>Think</u>: "positive terms, getting smaller so s_n must $\to 0$."

<u>Ex</u> $a_n = \frac{1}{2^n}$ $\quad \frac{a_{n+1}}{a_n} = \frac{1/2^{n+1}}{1/2^n} = \frac{2^n}{2^{n+1}} = \frac{1}{2} < 1$

conv's.

$b_n = \frac{2^n}{n!}$ $\quad \frac{b_{n+1}}{b_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \to 0 < 1$

conv's.

$c_n = \frac{1}{n}$ $\quad \frac{c_{n+1}}{c_n} = \frac{1}{n+1} \cdot \frac{n}{1} = \frac{n}{n+1} \to 1.$

⚠ Thus Thm 17.7 says <u>nothing</u> about convergence of c_n. (Although we know from §16 that it <u>does</u> converge)

Finally, infinite limits

Def s_n diverges to $+\infty$ if: instead of getting close to a limit s, the #s s_n eventually gets larger than any chosen #:

$$\forall M \in \mathbb{R}\ \exists N \text{ s.t. } n > N \Rightarrow s_n > M.$$

Similarly, $s_n \to -\infty$ (diverges!) if

$$\forall m \in \mathbb{R}\ \exists N \text{ s.t. } n > N \Rightarrow s_n < m.$$

Ex $(s_n) = (n) = (1, 2, 3, 4, \ldots)$

Let $M \in \mathbb{R}$ be given. Set $N = M$.

Then $n > N \Rightarrow \underbrace{s_n}_{=n} > M$.

$(t_n) = (-n^2) = (-1, -4, -9, -16, \ldots)$

Given $M \in \mathbb{R}$, we want to show $-n^2$ eventually less than M.

equiv: $-n^2 < M$

$n^2 > -M = |M|$ assuming $M < 0$.

set $N = \sqrt{|M|}$

then $n > N \Rightarrow n^2 > |M|,\ -n^2 < -|M|$.

Thm 17.12 (Comparison Thm for ∞ limits)

Suppose $s_n \leq t_n$ $\forall n \in \mathbb{N}$.

(a) $s_n \to \infty \Rightarrow t_n \to \infty$

(b) $t_n \to -\infty \Rightarrow s_n \to -\infty$

You Read:

Thm 17.13 $s_n > 0$, $s_n \to +\infty \Rightarrow \frac{1}{s_n} \to 0$.

(Think: $s_n = n$, $s_n \to \infty$,
$1/s_n = 1/n = 0$.)