§32 Infinite Series

Recall: given (a_n), then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots\cdots$$

$$S_1 \quad S_2 \quad S_3 \quad S_4$$

A sum of the terms in a sequence is a series; above we have an infinite series.

Q: when can we say an inf. series has a value?

A: $\sum a_n$ has an associated sequence of partial (or truncated) sums.

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

If (and only if) $s_n \to s$ may we say

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots = s.$$

Otherwise the series diverges and does not equal a real number.

⚠ remember these warnings:

① $a_1 + a_2 + a_3 + \cdots$ has no arithmetical value unless $\sum a_n$ converges. So

$\underbrace{a_1 + a_2 + a_3 + \cdots}_{s}$ is really $\lim \underbrace{(a_1 + \cdots + a_n)}_{= \lim s_n}$

② Think of $a_1 + a_2 + a_3 + \cdots$ as one object. Don't apply laws of arithmetic to these infinite sums; don't rearrange, regroup, etc.

Exercise 33.16

Ex Could write:

$$(1-1)+(1-1)+(1-1)+\cdots$$

$$= 0+0+0+\cdots$$

$$= 0$$

As long as it's understood that all parentheses are evaluated before the infinite sum. We may not rearrange:

$$0 = ((1-1)+(1-1)+\cdots$$
$$= 1+(-1+1)+(-1+1)+\cdots$$
$$= 1+0+0+\cdots$$
$$= 1.$$

Ex

$$\sum_{n=0}^{\infty} r^n = 1+r+r^2+\cdots$$

check on HW!

$$1+r+r^2+\cdots+r^n = \frac{1-r^{n+1}}{1-r} \longrightarrow \frac{1}{1-r}$$

$\underline{\text{if}}$ $|r|<1$.

More gen'lly: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, $|r|<1$ (HW)

Ex Evaluate

1st term of full series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\right) - \left(\frac{1}{2}\right)^0$$

$= 2-1 = 1$

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} = \frac{1}{1/2} = 2.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2}{3^n} = \sum_{n=0}^{\infty} 2\left(-\frac{1}{3}\right)^n = \frac{2}{1+1/3} = \frac{2}{4/3}$$

$$= \frac{6}{4} = \frac{3}{2}.$$

$$\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n \overset{\infty}{\underset{}{=}} \neq \frac{1}{1-3/2} = \frac{1}{-1/2} = -2.$$

$|r| = \frac{3}{2} > 1.$

The thms in §§17-18 all apply to series (by applying them to seq. of partial sums). For example:

Thm 32.6 (Paraphrased)

$\sum a_n$ converges $\Leftrightarrow$ $s_n = \sum_{k=1}^{n} a_k$ Cauchy

$\Leftrightarrow$ s_n converges $\Leftrightarrow$

Thm 32.4 If $\sum a_n = s$, $\sum b_n = t$, $k \in \mathbb{R}$ $\Rightarrow$

(a) $\sum (a_n + b_n) = s + t$.

(b) $\sum k a_n = k \cdot s$.

Proof of (a) (Sketch)

$\sum a_n = s$ $\Leftrightarrow$ partial sums $s_n \to s$.

$\sum b_n = t$ $\Leftrightarrow$ $t_n = b_1 + b_2 + \dots + b_n \to t$.

$s_n + t_n =$ seq of part. sums of $\sum a_n + b_n$

By Thm 17.1, $s_n + t_n \to s + t$

$\Rightarrow \sum (a_n + b_n) = s + t$.

⚠ $\Leftarrow$ in last thm not true.

Ex $a_n = 1$, $b_n = -1$, $c_n = a_n + b_n = 0$

$$\sum c_n = 0 + 0 + 0 + \cdots = 0.$$

$$\sum a_n = 1 + 1 + 1 + \cdots = +\infty$$

$$\sum b_n = -1 - 1 - 1 - 1 - \cdots = -\infty$$

Do NOT split up a series

$$\sum (a_n + b_n) = \sum a_n + \sum b_n$$

unless you know the seperate pieces converge!

Example The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

$s_1 = 1$

$s_2 = 3/2$

$s_3 = 11/6$

$s_4 = 25/12$.

$\vdots$

Does $\sum \frac{1}{n}$ converge? ($\Leftrightarrow$ does s_n converge?)

We know s_n converges $\iff$ s_n Cauchy:

$$\forall \varepsilon > 0 \ \exists N \text{ s.t. } \forall n, m > N$$
$$|s_n - s_m| < \varepsilon.$$

We'll show s_n **not** Cauchy, so s_n diverges (hence $\sum \frac{1}{n}$ too!)...

Suppose $m > n$: (so $\frac{1}{m} < \frac{1}{n}, \frac{1}{n+1}$, etc)

$$s_m - s_n = \left(\underline{1 + \tfrac{1}{2} + \dots + \tfrac{1}{n}} + \tfrac{1}{n+1} + \dots + \tfrac{1}{m}\right) - \left(\underline{1 + \dots + \tfrac{1}{n}}\right)$$

$$= \underbrace{\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{m}}_{m-n}$$

$$> \frac{1}{m} + \frac{1}{m} + \dots + \frac{1}{m}$$

So $s_m - s_n > \frac{(m-n)}{m} = 1 - \frac{n}{m}$

If $m = 2n$, $s_m - s_n > 1 - \frac{n}{2n} = \frac{1}{2}$

Let $\varepsilon = \frac{1}{2}$, N any nat'l #. Choose and $n, m > N$ with $m = 2n$. Then $|s_m - s_n| > \frac{1}{2} = \varepsilon$

Warmup Problem

Prove: If $\sum a_n = s$ and $c \in \mathbb{R}$, then

$$\sum c a_n = c a_1 + c a_2 + \cdots = c \cdot s.$$

Can't write:

$$(c a_1 + c a_2 + \cdots) = c\underbrace{(a_1 + a_2 + \cdots)}_{s} = c \cdot s.$$

Better

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$s_n = a_1 + \cdots + a_n$$

Partial sums of $\sum c a_n$ are

$$c a_1 = c s_1$$
$$c a_1 + c a_2 = c(a_1 + a_2) = c s_2$$
$$c a_1 + c a_2 + c a_3 = c(a_1 + a_2 + a_3) = c s_3$$
$$\vdots$$
$$c \cdot s_n$$

Thm 17.1 says $\lim (c s_n) = c \lim s_n = c \cdot s$

$\Rightarrow \sum c a_n = cs.$

Last Week

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \quad \text{diverges!}$$

We showed this by proving we can never force s_n, s_m to be "close" – $\boxed{s_{2n} - s_n \geq \frac{1}{2}}$

So (s_n) not Cauchy $\Leftrightarrow$ s_n diverges

But s_n increasing:

$$s_{n+1} = s_n + \frac{1}{n+1} > s_n$$

$\Rightarrow$ s_n cannot be bounded

If it were, MCT would say is converges (and it doesn't)

$\Rightarrow$ $s_n \to \infty$ which means:

$$\sum \frac{1}{n} = +\infty$$

In other words, given <u>any</u> $M \in \mathbb{R}$

$\exists$ N such that:

$$s_N = a_1 + a_2 + a_3 + \cdots + a_N > M$$

$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ increases so slowly that these #'s are <u>enormous</u>.

Tool: $1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \gamma + \ln n, \quad \gamma \approx 0.577\ldots$

n	s_n	$\gamma + \ln n$
1	1	.577
2	1.5	1.27
10	2.93	2.88
100	5.187	5.182
1000	7.485	7.48497

So for $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} > M$,

solve $\ln n + \gamma > M$.

$\ln n > M - \gamma$

$n > e^{M - \gamma}$

$M = 100 \Rightarrow n > 1.5 \times 10^{43}$

$M = 1000 \Rightarrow n > 10^{434}$

$M = 1\,000\,000 \Rightarrow n > 10^{434\,294}$

Does anything diverge *more* slowly? Yes!

Examples from ≈ 5615/5616:

$$\sum_{p \text{ prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = +\infty$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$$

$$= \zeta(3) \approx 1.20205\ldots$$

(irrational...)

(Apery)

Let's consider graphical representations of infinite series – much like "geometric fractions" in grade school.

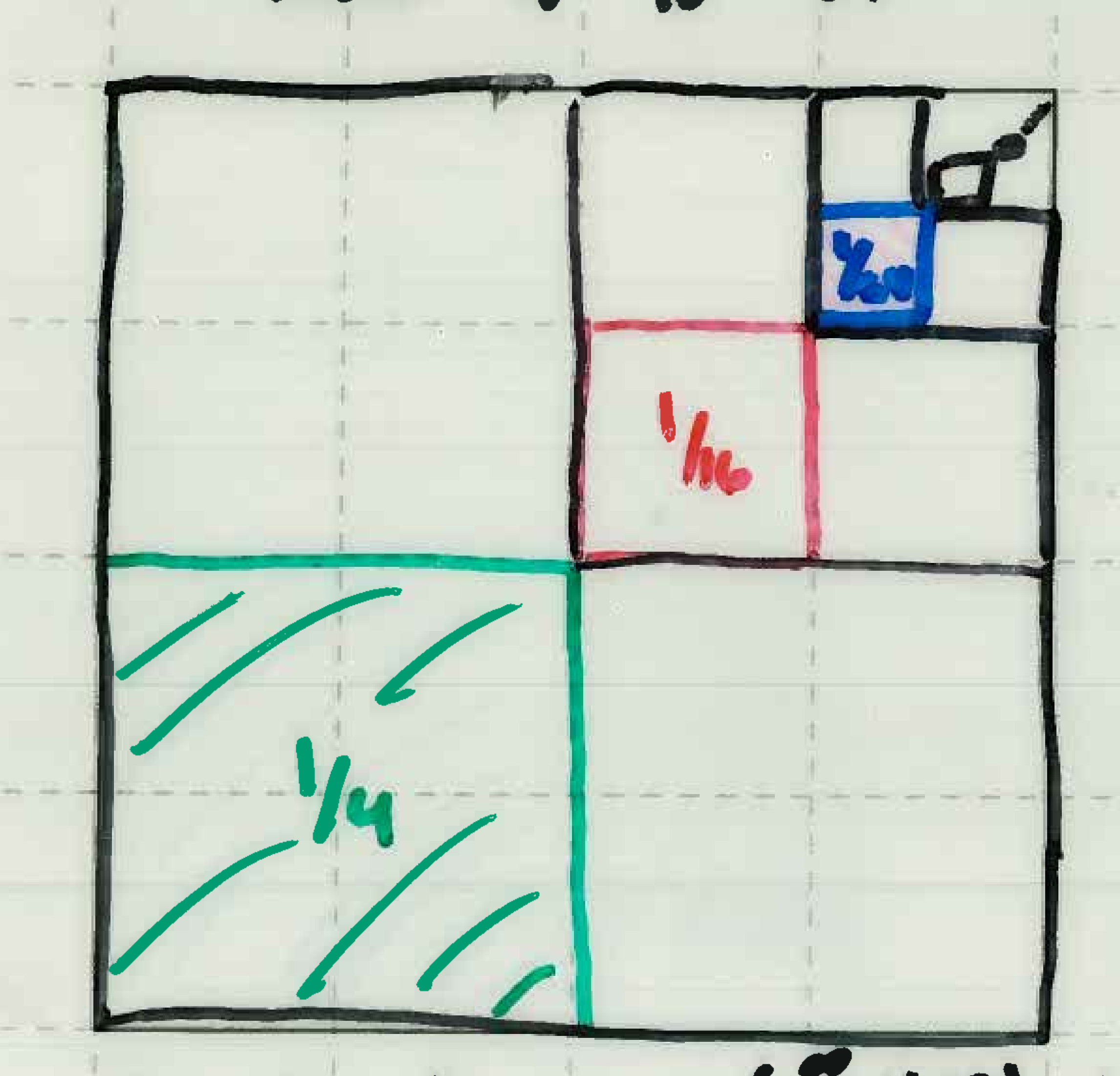
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=1$$

b/c every pt will be shaded.
(any pt in interior eventually shaded)

$$\left(\sum_{n=0}\left(\frac{1}{2}\right)^n\right)-\frac{1}{2^0}=\frac{1}{1-1/2}-1=2-1$$
$$=1.$$

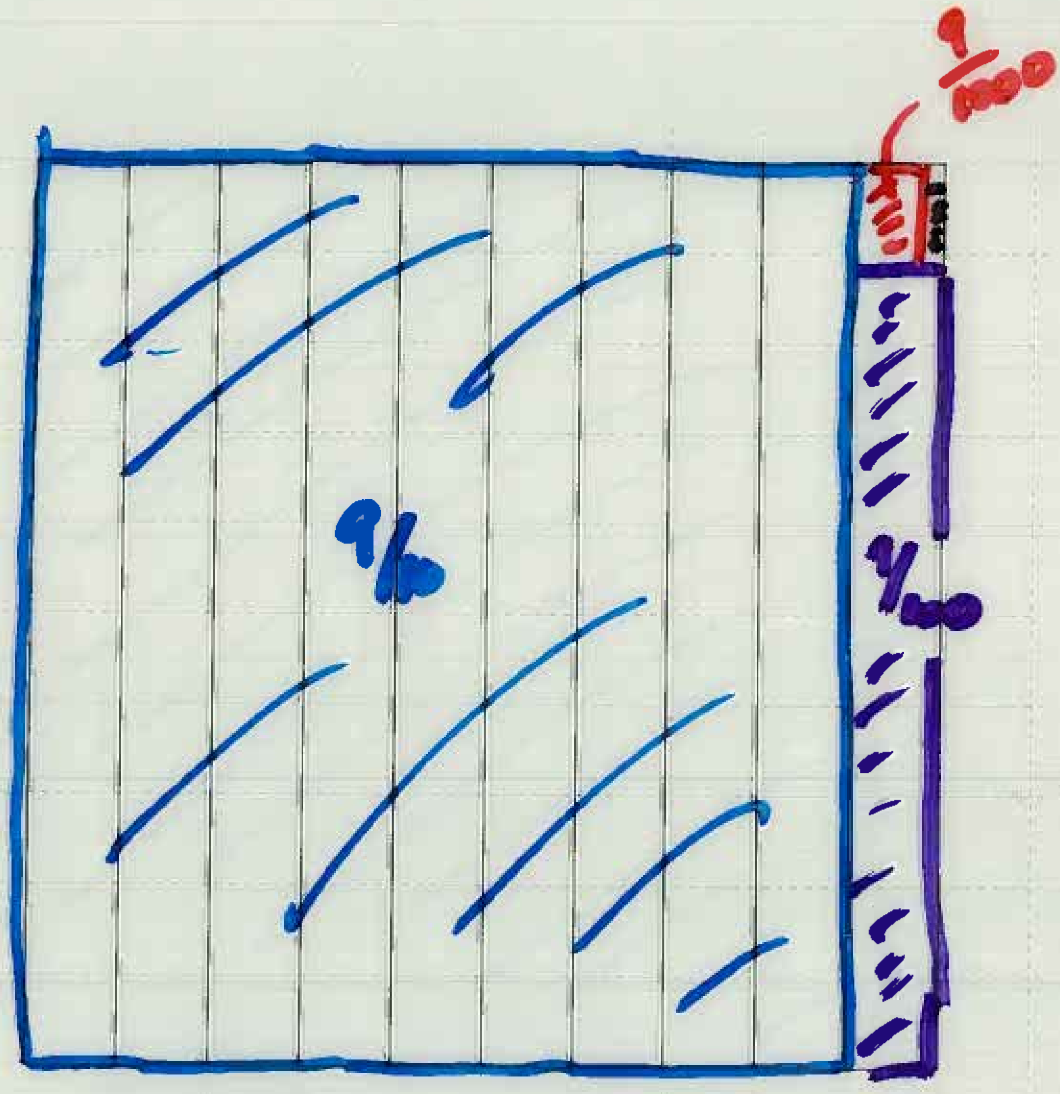
White square is decomposed into shapes, $\frac{1}{3}$ of each is shaded.

Thus $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{3}$.



$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \left(\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n\right) - 1$$

$$= \sum_{n=0}^{\infty} \underset{a}{\frac{1}{4}} \underset{r}{\left(\frac{1}{4}\right)}^n = \frac{1/4}{1 - 1/4} = \frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}.$$



$$\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots = 1$$

$$0.999999\ldots = 1.$$

$$\sum \frac{9}{10}\cdot\left(\frac{1}{10}\right)^n = \frac{9/10}{1-1/10} = \frac{9}{10}\cdot\frac{10}{9} = 1.$$

Back to the book - your text has geometric explanations of the convergence of geometric series

$1 + r + r^2 + \cdots$

one where $0 \le r < 1$, one where $-1 < r < 0$.

Uses analytic geometry, similarity of Δ

Final wrapup:

<u>Thm 32.5</u> If $\sum a_n$ converges, then $a_n \to 0$

<u>Converse not true!!!</u> ($\sum \frac{1}{n} = \infty$

Intuitively, if $a_n \to 1$ (or some other number),

$$\sum a_n \approx 1 + 1 + 1 + 1 + 1 + 1 + 1 + \cdots = \infty$$