Math 3283W - Exam 2 Review Worksheet[1]

§2.3 - Functions

1. For a function $f : A \to B$, the function $g : B \to A$ is called

   a **left inverse** for f if $g \circ f$ is the identity on A (i.e., $g \circ f = \mathrm{id}_A$)

   and

   a **right inverse** for f if $f \circ g$ is the identity on B (i.e., $f \circ g = \mathrm{id}_B$).

   (a) Prove: f has a left inverse if and only if f is injective.

   (b) Prove: f has a right inverse if and only if f is surjective.

2. Let A, B, C be sets such that $C \subseteq B$. Prove that $f^{-1}(B \setminus C) = A \setminus f^{-1}(C)$.

3. Let A, B, C be sets such that $C \subseteq B$.

   (a) Prove: if f is surjective, then $f(f^{-1}(C)) = C$.

   (b) Give an example of an function f such that $f(f^{-1}(C)) \subsetneq C$.

§2.4 - Cardinality

1. For S and T sets, state the definition of $|S| \leq |T|$.

2. Give an example of a set $S \subsetneq \mathbb{R}$ such that

   (a) S is denumerable.

   (b) S is uncountable.

   Prove your claims. You may use the fact that $\mathbb{R}$ is uncountable.

3. Recall, the power set $\mathcal{P}(S)$ of a set S is defined by the property

$$A \in \mathcal{P}(S) \iff A \subseteq S.$$

   (a) What is $\mathcal{P}(\{a, b, c, d\})$?

   (b) Prove: for every set S, $|S| \leq |\mathcal{P}(S)|$. (In fact, $|S| < |\mathcal{P}(S)|$, though you are not being asked to prove this stronger statement.)

4. Prove: if S is denumerable, then there exists a proper subset T of S such that $S \sim T$.

§3.1 - Induction

1. Prove using induction: $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$.

[1] This worksheet is far from inclusive. Do **not** assume that doing only these problems will fully prepare you for Exam 2.

2. Prove using induction: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{N}$.

3. Prove using (1) and (2): $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ for all $n \in \mathbb{N}$.

§3.2 - Fields

1. Prove that $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$.

2. Prove that $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

§3.3 - Completeness

1. State the definition of supremum and infimum for a set S.

2. For each set listed below, determine whether the supremum and/or infimum exist. If so, determine the value of the supremum and/or infimum. If not, state why.

   (a) $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$
   (b) $(-\infty, 4)$
   (c) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

3. Let S and T be bounded, nonempty sets such that $S \subseteq T \subseteq \mathbb{R}$. Prove

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

§3.4 - Topology

1. Consider the following subsets of $\mathbb{R}$:

   (a) $\{\pi\}$
   (b) $\mathbb{Q}$
   (c) $\bigcap_{n=1}^{\infty}(0, \frac{1}{n})$
   (d) $(x, y]$ where $x < y$ are real numbers.

   For each set, determine and prove:

   - Is the set open, closed, both, or neither?
   - The boundary and interior of the set.

2. Prove:

   (a) A union of open sets is open.
   (b) An intersection of finitely many open sets is open.
   (c) An intersection of closed sets is closed.
   (d) A union of finitely many closed sets is closed.

3. Give an example of an infinite intersection of open sets which is not open, and an infinite union of closed sets which is not closed.

4. Let $S \subseteq \mathbb{R}$ be nonempty, open, and bounded above. Prove that $\sup S \notin U$.