Useful Consequence / Explanation of Terminology

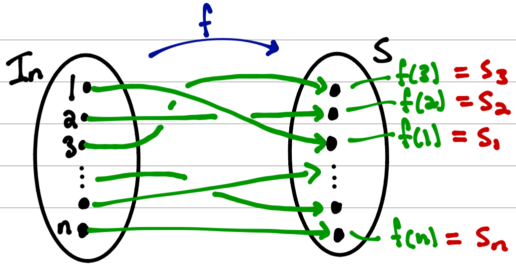
Countable sets can be "listed in order."

S finite: write $S = \{s_1, s_2, s_3, \dots, s_n\}$

S denumerable: write $S = \{s_1, s_2, s_3, \dots\}$

⚠ No canonical "1^{st} element", 2^{nd}, etc — many orders possible.

Ex $I_n \sim S$.



Ex For our example $\mathbb{N} \to \mathbb{Z}$,

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \dots\}$$

Thm (Ex 2.4.11) Let S, T be countable. Then $S \cup T$ is too.

Pf $\exists$ three cases: (1) Both S, T are finite

(2) One is finite, one is denumerable

(3) Both are denumerable.

⚠ Left to you: what the sets overlap?

Case 1 $S \sim I_n$, $T \sim I_k$ $\exists$ bij'ns $h: I_n \to S$

$g: I_k \to T$

We want: bij'n from I_{n+k} to $S \cup T$.

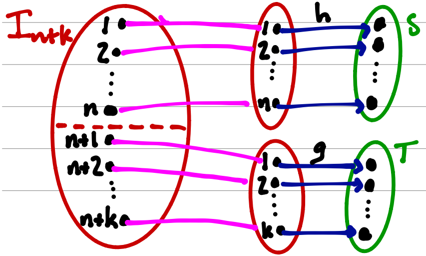
Thm (Ex 2.4.11) Let S, T be countable. Then $S \cup T$ is too.

Pf $\exists$ three cases: ① S, T both finite
② One is finite, one is denumerable
③ Both are denumerable.

Left for you: What if there's overlap, $S \cap T \neq \emptyset$?

Case 1 $S \sim I_n$, $T \sim I_k$, so $\exists$ bij'ns $h: I_n \to S$, $g: I_k \to T$

We want a bij'n $f: I_{n+k} \to S \cup T$



$$f(p) = \begin{cases} h(p), & p \leq n \\ g(p-n), & p > n \end{cases}$$

Case 2 WLOG (Without Loss of Generality), assume S is finite, T denumerable.

Construct bij'n $f: \mathbb{N} \to S \cup T$ as follows:

$$S \cup T = \{ s_1, s_2, \ldots, s_n, t_1, t_2, t_3, t_4, \ldots \}$$
$$f(1), f(2), \ldots, f(n), f(n+1), f(n+2), \ldots$$

Case 3 Suppose $f: \mathbb{N} \to S$, $g: \mathbb{N} \to T$ are bij'ns

Define $h: \mathbb{N} \to S \cup T$ by

$$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right), & n \text{ odd} \\ g\left(\frac{n}{2}\right), & n \text{ even} \end{cases}$$

Corollary $\mathbb{Q}$ is countable.

Pf $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^-$.

$$\mathbb{N} \sim \mathbb{Q}^+ \sim (\mathbb{Q}^+ \cup \mathbb{Q}^-)$$

Thm 2.4.3 Any subset of a countable set S is countable.

"Proof by Example." Let $P = \{\text{set of primes}\} \subseteq \mathbb{N}$.

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, \ldots\}$$

bij'n $f: \mathbb{N} \to P$ given by: $f(1) = 2$

$f(2) = 3$

$f(n) = n^{th}$ prime on list.

Thm 2.4.12 $\mathbb{R}$ is uncountable
(part of your intellectual heritage!)

Pf: Using CP of previous thm, we'll show $(0,1)$ is uncountable, which implies $\mathbb{R}$ is uncountable.

Assume $(0,1)$ is countable, so we can list its elts in order:

$$\begin{aligned}
x_1 &= 0.\,x_{11}\,x_{12}\,x_{13}\,x_{14}\,x_{15}\cdots\\
x_2 &= 0.\,x_{21}\,x_{22}\,x_{23}\,x_{24}\,x_{25}\cdots\\
x_3 &= 0.\,x_{31}\,x_{32}\,x_{33}\,x_{34}\,x_{35}\cdots\\
x_4 &= 0.\,x_{41}\,x_{42}\,x_{43}\,x_{44}\,x_{45}\cdots\\
x_5 &= 0.\,x_{51}\,x_{52}\,x_{53}\,x_{54}\,x_{55}\cdots
\end{aligned}$$

⚠ To make decimal expansion unique, write $0.4\overline{999}$ as 0.5, and so on.

$$x_1 = 0.\, x_{11}\, x_{12}\, x_{13}\, x_{14}\, x_{15} \cdots$$
$$x_2 = 0.\, x_{21}\, x_{22}\, x_{23}\, x_{24}\, x_{25} \cdots$$
$$x_3 = 0.\, x_{31}\, x_{32}\, x_{33}\, x_{34}\, x_{35} \cdots$$
$$x_4 = 0.\, x_{41}\, x_{42}\, x_{43}\, x_{44}\, x_{45} \cdots$$
$$x_5 = 0.\, x_{51}\, x_{52}\, x_{53}\, x_{54}\, x_{55} \cdots$$

<u>Goal</u>: find a number $b = 0.b_1 b_2 b_3 b_4 \cdots$ which is <u>not</u> on list.

($\Rightarrow$ $(0,1)$ <u>not</u> ct'ble $\Rightarrow$ $\mathbb{R}$ uncountable)

<u>Define</u> b by $b_n = \begin{cases} 2, & x_{nn} \neq 2 \\ 3, & x_{nn} = 2 \end{cases}$

By construction, b cannot be on list.

(its n^{th} digit is different than x_{nn}, so $b \neq x_n$)

Ex if our list looked like this:

$$x_1 = 0.123123123\ldots$$
$$x_2 = 0.333333333\ldots$$
$$x_3 = 0.222222\ldots$$
$$x_4 = 0.98765\ldots$$

$$b = 0.2232$$