Chapter 3: The Real Numbers

§3.1 $\mathbb{N}$ (?!) and induction.

$\mathbb{N}$ provides a nice intro to properties of number systems and sets. For example:

Axiom 3.1.1 $\mathbb{N}$ is *well ordered*, which means:

If $\emptyset \neq S \subseteq \mathbb{N}$, then it has "least" elt, i.e.

$$\exists\, k \in S \text{ such that } k \leq m \;\; \forall\, m \in S.$$

Ex $S = \{10, 9, 8, 100, 99, 98, 1000, 999, 998, \ldots\} \subseteq \mathbb{N}$

$k = 8 \leq m \quad \forall m \in S.$

Aside #1 Could look at the rest of the elts in S (if there are any), choose the least of those, repeat.

$\Rightarrow$ can write S in numerical order.

$S = \{8, 9, 10, 98, 99, 100, 998, 999, 100, \ldots\}$

Aside #2 Can every set be well-ordered?

What's the smallest/minimal elt in $(0,1)$? or $\mathbb{R}$?

Well Ordering "Theorem" Every set can be well-ordered (with respect to some order — not necessarily $<$, $\leq$, etc)

Hmm... believable? Equivalent to:

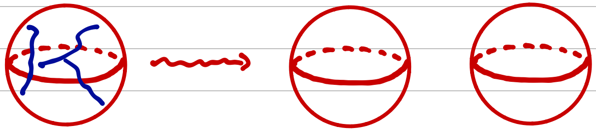
Axion of Choice Given any infinite collection of bins (set) we can choose one object (elt) from each.

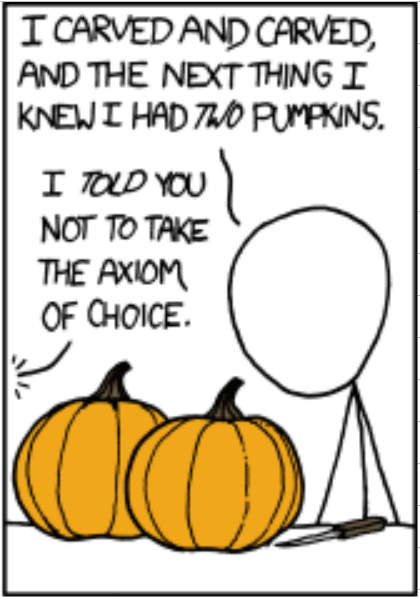
Seems reasonable.... but has weird consequences, like WOT above, and....

Banach-Tarski Paradox

A sphere in $\mathbb{R}^3$ can be cut into a finite number of pieces, which can be rearranged and glued back together...

into <u>two</u> identical copies of original sphere (!!?!)




I CARVED AND CARVED, AND THE NEXT THING I KNEW I HAD TWO PUMPKINS.
I TOLD YOU NOT TO TAKE THE AXIOM OF CHOICE.

Back to Reality

Thm 3.1.2 (Proof by Induction)

Let $P(n)$ be a stmt which is T/F for each n. If

base → (a) $P(1)$ is true

induction step → (b) $\forall k \in \mathbb{N}$, if $P(k)$ true then $P(k+1)$ true

Then $P(n)$ true for all $n \in \mathbb{N}$.

idea: $P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \cdots$ (infinite ladder)

[could start other than $n=1$]

Pf: Let $S = \{k \mid P(k) \text{ is false}\} \subseteq \mathbb{N}$. If $S = \emptyset$, then $P(n)$ true $\forall n$, we're done. If $S \neq \emptyset$, it has a least elt m [by well ordering]. By (a) [base case] $m \neq 1$, so $m-1 \in \mathbb{N}$. $P(m-1)$ true, so (b) $\Rightarrow$ $P(m)$ true $\lightning$.

Obligatory Historical Example

Prove: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

Check: $P(1)$: $1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$ ✓

$P(4)$: $1+2+3+4 = 10.$

$\frac{4(5)}{2} = 10.$ ✓

$P(1000)$: $1+2+3+4+\cdots+1000$

$1000+999+998+997 \cdots + 1$

$1001+1001+1001+\cdots+1001 = 1000(1001)$

Answer: $\frac{1}{2}(1000 \cdot 1001)$

Inductive Proof $1+2+3+\cdots+n = \frac{n(n+1)}{2}$. : $P(n)$

Base case: checked $P(1)$ above

Induction step: Assume $P(n)$ is true, $1+\cdots+n = \frac{n(n+1)}{2}$, and we want to show $P(n+1)$ is true:

$$1+2+3+\cdots+n+(n+1) = \underbrace{\left[1+2+3+\cdots+n\right]}_{\text{use assumption}} + (n+1)$$

$$= \left[\frac{n(n+1)}{2}\right] + (n+1)$$

$$= \frac{(n^2+n)+2n+2}{2}$$

$$= \frac{n^2+3n+2}{2}$$

$$= \frac{(n+1)(n+2)}{2}, \text{ as desired.}$$

☆ Often use $P(k)$, $P(k+1)$ instead of n here.

⚠ What's wrong with this "inductive proof"?

Base case: $n=1$: $1 = \frac{1(2)}{2}$ ✓

Inductive Step: Assume true for n, prove for $n+1$:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$1+2+3+\cdots+n+(n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$1+2+3+\cdots+n+(n+1) = \frac{n^2+n+2n+2}{2}$$

$$1+2+3+\cdots+n+(n+1) = \frac{n^2+3n+2}{2}$$

$$1+2+3+\cdots+n+(n+1) = \frac{(n+1)(n+2)}{2},$$ as desired.

⚠ What's wrong with this "inductive proof"? (V2)

Base case: $n=1$: $1 = \frac{1(2)}{2}$ ✓

Inductive Step: Assume true for n, prove for $n+1$:

$$1+2+3+\cdots+n+(n+1) = \frac{(n+1)(n+2)}{2}$$

$$1+2+3+\cdots+n = \frac{(n+1)(n+2)}{2} - (n+1)$$

$$1+2+3+\cdots+n = \frac{(n^2+3n+2)-2n-2}{2}$$

$$1+2+3+\cdots+n = \frac{n^2+n}{2}$$

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$, which is true by assumption.

General Guidelines:

1. Don't work with both sides of an equality at once

2. Related: don't say things are equal until you know they're equal.

avoid: (eqn that) = (we want)

$\sim\sim = \sim\sim$

$\sim\sim = \sim\sim$

(something) = (we know) ✓