§3.2 Ordered Fields

You should read this section, but you're not responsible for all of it.

Focus on the def$^{\underline{s}}$ we discuss in lecture, the few thms we'll mention, and any HW assigned.

The rest is "ambiance" showing just how much work goes into constructing algebra.

For more: fields – take modern/abstract algebra

ordered fields – take Math 5615/6.
(Real Analysis)

$\mathbb{R}$ is a complete, ordered field, meaning:

Complete: will be described in next section. Essentially means $\mathbb{R}$ has no "holes."

Ex in $\mathbb{Z}$, no #'s b/w 0 and 1.

Ex in $\mathbb{Q}$, there is no # q s.t. $q^2=2$
But $\sqrt{2} \in \mathbb{R}$.

ordered Given any $x, y \in \mathbb{R}$ we can put them in order: $x<y$, $y \leq x$, etc.

field means we can do all standard arithmetic
$+$, $-$, $\cdot$, $\div$ (but not by 0)

Addition axioms

A1: $\forall x, y \in \mathbb{R},\ x + y \in \mathbb{R}$.
If $x = w$ and $y = z$, then $x + y = w + z$.
A3: $\forall x, y, z \in \mathbf{R}, x + (y + z) = (x + y) + z$.
A4: $\exists ! 0 \ni \forall x \in \mathbf{R}, x + 0 = x$.
A5: $\forall x \in \mathbf{R}, \exists ! -x \ni x + (-x) = 0$.

A2: $\forall x, y \in \mathbf{R},\ x + y = y + x$.

A1, A3, A4, A5 make $\mathbf{R}$ a *group* under addition.
If **A2** is also true, we say that the group is *commutative* (or *abelian*).

Multiplication axioms

M1: $\forall x, y \in \mathbb{R}$, $x \cdot y \in \mathbb{R}$.
If $x = w$ and $y = z$, then $x \cdot y = w \cdot z$.
M3: $\forall x, y, z \in \mathbf{R}, x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
M4: $\exists! 1 \ni 1 \neq 0$ and $\forall x \in \mathbf{R}, x \cdot 1 = x$.
M5: $\forall x \in \mathbf{R}$ with $x \neq 0, \exists! x^{-1} \ni x \cdot x^{-1} = 1$.

M2: $\forall x, y \in \mathbf{R}$, $x \cdot y = y \cdot x$.

And one axiom governing how the two operations interact:

DL: $\forall x, y, z \in \mathbf{R}, x \cdot (y + z) = x \cdot y + x \cdot z$.

Order axioms

O1: $\forall x, y \in \mathbf{R}$, exactly one holds: $x = y, x < y, y < x$.

O2: If $x < y$ and $y < z$, then $x < z$.

O3: If $x < y$, then $x + z < y + z$.

O4: If $x < y$ and $z > 0$, then $x \cdot z < y \cdot z$.

Examples

$\mathbb{N}$: $\forall m, n \in \mathbb{N}$, $m+n \in \mathbb{N}$ (we can add)
$10 - 15 = -5 \in \mathbb{Z}$, not $\mathbb{N}$. (can't do $-$).
Can multiply, not divide in gen'l.

$\mathbb{Z}$: we can add, subtract, or multiply two integers and get an integer. BUT...

in gen'l, can't divide: $\frac{1}{2} \notin \mathbb{Z}$.

$\mathbb{Q}$: can add, subtract, multiply, divide (by non-zero) fractions and get fractions.

field (ordered, even)

Thm 3.2.2 Let $x, y, z \in \mathbb{R}$

(A4) $\exists!\ 0 \in \mathbb{R} \ni \forall x \in \mathbb{R},\ x + 0 = x$

(DL) $x(y+z) = xy + xz$

(A2) $x + y = y + x$

(a) $x + z = y + z \Rightarrow x = y$

(b) $x \cdot 0 = 0$

(c) $-0 = 0$

(d) $(-1) \cdot x = -x$

(e) $xy = 0 \Leftrightarrow x = 0$ or $y = 0$

(f) $x < y \Leftrightarrow -y < -x$

(g) $x < y$ and $z < 0$
$\Longrightarrow zy < zx$

Pf of (b)

$x \cdot 0 = x(0+0)$ (A4)

$x \cdot 0 = x \cdot 0 + x \cdot 0$ (DL)

$x \cdot 0 + 0 = x \cdot 0 + x \cdot 0$ (A4)

$0 + x \cdot 0 = x \cdot 0 + x \cdot 0$ (A2)

$0 = x \cdot 0$ (part (a))

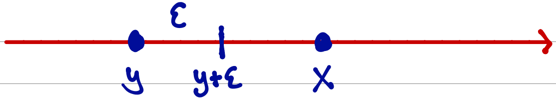
Thm 3.2.8 If $x \le y + \varepsilon \ \forall \varepsilon > 0$, then $x \le y$.

Remark: Think "limits". $y + \varepsilon$ for all $\varepsilon > 0$ is funny way to say $\lim_{h \to 0^+} y + h$.

Thm says: $\lim_{h \to 0^+} x \le \lim_{h \to 0^+} y + h \Rightarrow x \le y$.

CP: if $y < x$ then $\exists \varepsilon > 0 \ni y + \varepsilon < x$.

Pf of: if $y<x$ then $\exists\ \varepsilon>0 \ni x > y+\varepsilon$



Could use $y+\varepsilon = \frac{x+y}{2} \Rightarrow \varepsilon = \frac{x+y}{2} - y = \frac{x-y}{2}$

check: $y+\varepsilon = y + \frac{x-y}{2}$

$$= \frac{2y+x-y}{2}$$

$$= \frac{x+y}{2}$$

$$< \frac{x+x}{2} \quad \text{(b/c } x>y\text{)}$$

$$= x$$

Thm 3.2.10

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

(a) $|x| \geq 0$

(b) $|x| \leq a \iff -a \leq x \leq a$

(c) $|xy| = |x| \cdot |y|$

(d) $\underbrace{|x+y| \leq |x| + |y|}_{\text{triangle inequality}}$

clearer w/ vectors: