§ The Completeness Axiom

Or: How does $\mathbb{R}$ differ from $\mathbb{Q}$?

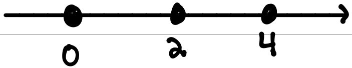
Major Concepts

1. Upper, lower bounds
2. sup, inf:
   - defs
   - finding $\underbrace{\sup S, \inf S}$
   - proving your answers are correct.
3. Completeness
4. Density of $\mathbb{Q}$
5. Archimedean Property.

Begin with bounds: Some subsets of $\mathbb{R}$ have minimum and/or maximum elements

- $m \in S$ is $\min S$ if $\forall s \in S,\ m \leq s$
- $M \in S$ is $\max S$ if $\forall s \in S,\ s \leq M$

Ex $S = \{0, 2, 4\}$

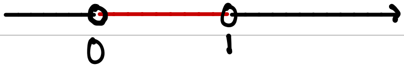


$\min S = 0$
$\max S = 4$

$S = [0, \infty)$

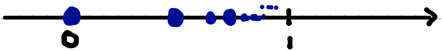
$\min S = 0$
no max

$S = (0, 1)$



no min, max

$S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$



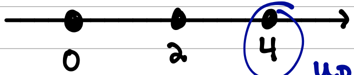
$\min S = 0$
no max

More generally: m is an <u>upper</u> bound for $S \subseteq \mathbb{R}$ if $\forall s \in S$, $s \le m$

m is a <u>lower</u> bound for $S \subseteq \mathbb{R}$ if $\forall s \in S$, $m \le s$

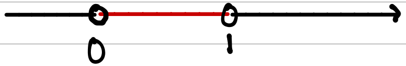
(need not be in S)

<u>Ex</u> $S = \{0, 2, 4\}$



upper bd <u>and</u> max

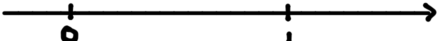
$S = (0, 1)$



0 is lower bd. So is $-1, -2, -3, \ldots$ $-\sqrt{2}, -17$

1 is upper bound. So is 5. or $\pi \ldots$

$S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$



Observations ① if m is upper bound for S, so is any # larger than m.

② m is lower bound $\Rightarrow$ so is any smaller #.

③ If an upper bd $m \in S$, then it's max S.

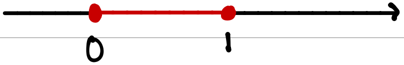
Def Let $\emptyset \neq S \subseteq \mathbb{R}$ be bounded above. (i.e. it has an upper bd)
The least upper bound is called supremum of S,

$$\sup S = \text{lub } S$$

Similarly, greatest lower bound of a set which is bdd below is infimum of S,

$$\inf S = \text{glb } S.$$

Ex $S = [0,1]$



$\sup S = 1$
$\inf S = 0$

$S = (0, \infty)$

$\inf S = 0$
no sup

$S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$

$\inf S = 0$
$\sup = 1$

So if $m = \sup S$,

(a) $m \geq s \ \forall s \in S$.

(b) $m' < m$ can't be an upper bound: $\exists s' \in S \ni s' > m'$

Similarly, if $m = \inf S$

(a) $m \leq s \ \forall s \in S$

(b) $m' > m$ can't be lower bound: $\exists s' < m'$

Ex (Old Exam Problem) $A = \left\{ \frac{n-1}{n+1} : n \in \mathbb{N} \right\}$

(a) List the elts of A.

$A = \left\{ \frac{0}{2}, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}, \ldots \right\} = \left\{ 0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \ldots \right\}$

(b) Show A is bounded above and below

$n \in \mathbb{N} \Rightarrow n-1, n+1$ both $\geq 0 \Rightarrow \frac{n-1}{n+1} \geq 0$, so A is bounded below by 0.

Also $n-1 < n+1 \Rightarrow \frac{n-1}{n+1} < 1$ (b/c $n+1 > 0$)

(c) Find $\ell = \inf A$, $m = \sup A$. Prove your answers are right.

$\ell = \inf A = 0$, $m = \sup A = 1$

$A = \left\{\frac{n-1}{n+1} : n \in \mathbb{N}\right\} = \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \dots\right\}$

Proof that $\inf A = 0$

First, 0 is a lower bound of A by part (b).

Now suppose $\exists\, \ell > 0$ which is a lower bound for A. This is a contradiction, b/c $0 \in A$, so ℓ cannot a lower bound of A.

Key: shorter b/c $0 \in A$, so $0 = \min A$

$A = \left\{ \frac{n-1}{n+1} : n \in \mathbb{N} \right\} = \left\{ 0, 1/3, 1/2, 3/5, 2/3, 5/7, \dots \right\}$

Proof that $\sup A = 1$

By part (b), 1 is upper bound: $a \le 1 \ \forall a \in A$.

Now suppose $\exists\, m < 1$ which is also an upper bd. We'll show this is impossible, so 1 is in fact the least upper bd.

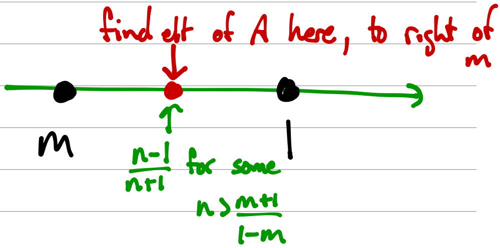
$\frac{n-1}{n+1} > m$

$n-1 > nm + m$

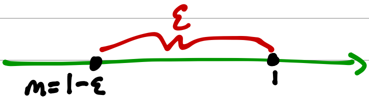
$n - nm > m + 1$

$n(1-m) > m+1$

$n > \frac{m+1}{1-m}$



⚠ Sometimes $m = 1 - \varepsilon$ for some small $\varepsilon > 0$



So solve $\dfrac{n-1}{n+1} > m = 1 - \varepsilon$

$$n - 1 > (n+1)(1-\varepsilon)$$

$$\vdots$$

$$n > \frac{\varepsilon + 2}{\varepsilon}$$

.... end result: $m = 1 - \varepsilon$ is not an upper bd. Hence 1 is least upper bound.

Completeness Axiom Every $\emptyset \neq S \subseteq \mathbb{R}$ which is bdd above has a least upper bound.

Notes ① $\mathbb{Q}$ not complete. Ex $S = \{x \in \mathbb{Q} : x^2 < 2\}$

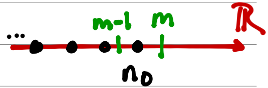
$0 \in S$, $1 \in S \Rightarrow S \neq \emptyset$. bdd above by 2 (1.5, 10, etc.)
BUT no least upper bd of S in $\mathbb{Q}$: $\sqrt{2} \notin \mathbb{Q}$

② Why not "Every $\emptyset \neq S \subseteq \mathbb{R}$ which is bdd below has inf?"

True – can prove using completeness Axiom

Thm 3.3.9 Archimedean Property of $\mathbb{R}$

$\mathbb{N}$ has no upper bound in $\mathbb{R}$.



Pf By contradiction. Suppose $\mathbb{N}$ is bdd above, $\mathbb{N} \neq \emptyset \Rightarrow \exists$ least upper bd, $m = \sup \mathbb{N}$. Then $m-1$ not upper bd $\Rightarrow \exists\, n_0 \in \mathbb{N}$, $n_0 > m-1$. Then $n_0 + 1 > m$, and $n_0 + 1 \in \mathbb{N}$, m not upper bd $\lightning$.

Thm 3.3.10 TFAE

(*) Archimedean Property

☆ (a) $\forall z \in \mathbb{R}$, $\exists\, n \ni n > z$.

(b) $\forall x > 0\ \forall y \in \mathbb{R}\ \exists\, n \in \mathbb{N} \ni nx > y$

☆ (c) $\forall x > 0\ \exists\, n \in \mathbb{N} \ni 0 < \frac{1}{n} < x$.