**Def** $(s_n)$ converges to $s \in \mathbb{R}$, written

$$s_n \to s \quad \underline{\text{OR}} \quad \lim_{n\to\infty} s_n = s \quad \underline{\text{OR}} \quad \lim s_n = s.$$

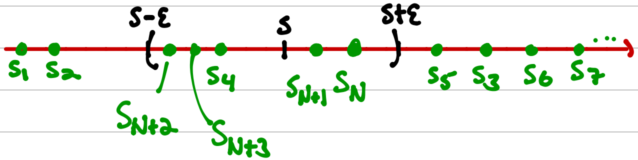
$$\forall \varepsilon > 0 \ \underline{\exists N} \in \mathbb{N} \ (\text{or } \mathbb{R}) \text{ s.t. } \underline{n > N} \Rightarrow |s_n - s| < \varepsilon.$$

**In words:** $s_n \to s$ if $\underset{\substack{n>N, \\ \text{Some } N}}{\underline{\text{eventually}}}$, every $s_n$ is $\underset{\forall \varepsilon > 0}{\underline{\text{(arbitrarily)}}}$ $\underset{|s_n - s| < \varepsilon}{\underline{\text{close to } s}}$.

⚠ Order matters! You don't choose $\varepsilon$. It's given to you, and your challenge is to prove you can (eventually) force $s_n$ within $\varepsilon$ of $s$.

# Visuals - via GeoGebra

Alternatively,



$s_N, s_{N+1}, s_{N+2}, \dots$ are all within $\varepsilon$ of $s$

$n \geq N \Rightarrow |s_n - s| < \varepsilon$ <u>or</u> $s_n \in (s-\varepsilon, s+\varepsilon)$

<u>or</u> $s_n \in N(s;\varepsilon)$

Note: at this point we're not worried about tools to find a limit. Just proving that a limit exists / is correct!

In general, to prove $s_n \to s$:

Step 1: Think / Algebra / Preparation. For some $\varepsilon > 0$, do algebra to find $N$ which makes rest of def$^{\underline{n}}$ work.

Step 2: Proof.

Let $\varepsilon > 0$. Choose $N =$ (magic # from above). Then if $n > N$,

$$|s_n - s| = \underbrace{\cdots (\text{algebra}) \cdots}_{\text{(substn)}} = \cdots < \varepsilon.$$

**Ex** $(s_n) = \left(\frac{n-1}{n}\right) = \left(\frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right)$ We suspect $s_n \to 1$

Think/Alg/Prep:

**Want:** figure out $n$ s.t. $|s_n - s| < \varepsilon$, i.e. $\left|\frac{n-1}{n} - 1\right| < \varepsilon$.

**Method 1**

$$-\varepsilon < \frac{n-1}{n} - 1 < \varepsilon$$

$$-\varepsilon < 1 - \frac{1}{n} - 1 < \varepsilon$$

$$\underbrace{-\varepsilon < -1/n}_{} < \varepsilon$$

always true; no useful info

$$\varepsilon > \frac{1}{n}$$

$n > 1/\varepsilon$ ← that's N.

**Method 2**

$$\left|\frac{n-1}{n} - \frac{n}{n}\right| < \varepsilon$$

$$\left|\frac{n-1-n}{n}\right| < \varepsilon$$

$$|-1/n| < \varepsilon$$

$$\frac{1}{n} < \varepsilon$$

$$1/\varepsilon < n$$

## Proof that $\frac{n-1}{n} \to 1$

Let $\varepsilon > 0$, and choose $N = 1/\varepsilon$. Then $n > N$ implies

$$\left|\frac{n-1}{n} - 1\right| = \cdots = \left|-\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{1/\varepsilon} = \varepsilon$$

$$\boxed{s_n \to s \text{ if } \forall \varepsilon > 0, \exists N \text{ s.t. } n > N \Rightarrow |s_n - s| < \varepsilon}$$

## Ex 4.1.6 Show $s_n = \frac{n^2+2n}{n^3-5} \to 0$

### Think/Algebra/Preparation

$\left|\frac{n^2+2n}{n^3-5} - 0\right| < \varepsilon$ messy to solve for $n >$ ___ $n^2+2n < n^3 \cdot \varepsilon - 5\varepsilon.$

Trick of the trade: find a "nicer" sequence which $\to 0$ and is bigger than ours:

$$0 \le \left|\frac{n^2+2n}{n^3-5} - 0\right| \le \left( \qquad \right) < \varepsilon$$

## Showing $\frac{n^2+2n}{n^3-5} \to 0$ via $\left|\frac{n^2+2n}{n^3-5} - 0\right| \leq (\quad\quad) < \varepsilon$

① We only care what eventually happens, so we can assume $n \geq 2$ so that

$$n^2+2n \geq 0 \quad \text{and} \quad n^3-5 \geq 0 \Rightarrow \left|\frac{n^2+2n}{n^3-5}\right| = \frac{n^2+2n}{n^3-5}$$

② Try to find a simpler (larger) seq. which bounds ours.

KEY $\frac{n^2+2n}{n^3-5} < \frac{p(n)}{q(n)}$ $\underline{\underline{\text{if}}}$ $p(n) > n^2+2n$, $q(n) < n^3-5$

Helps if $p, q$ have form $k \cdot n^{\text{power}}$, $\text{degree}(p) < \text{degree}(q)$

## Showing $\frac{n^2+2n}{n^3-5} \to 0$ via $\left|\frac{n^2+2n}{n^3-5} - 0\right| \leq \left(\frac{2n^2}{n^3/2}\right) < \varepsilon$

- For large enough $n$ (specifically, $n \geq 3$ )

$$n^2 > 2n \Rightarrow n^2+n^2 > n^2+2n$$
$$2n^2 > n^2+2n$$

$p(n) = 2 \cdot n^2$

- For large enough $n$ (specifically, $n \geq 3$ )

$n^3-5 > n^3$ never true!

What about $\frac{n^3}{2}$?

$$n^3-5 > \frac{n^3}{2} \Rightarrow n^3 > 10$$
$$n > \sqrt[3]{10}$$

$q(n) = \frac{1}{2} n^3$

## Showing $\frac{n^2+2n}{n^3-5} \to 0$ via $\left|\frac{n^2+2n}{n^3-5} - 0\right| \le \left( 4/n \right) < \varepsilon$

③ Find an appropriate N for our simpler sequence.

Figure out condition on $n$ s.t. $\left|\frac{4}{n} - 0\right| < \varepsilon$.

amounts to $\left|\frac{4}{n}\right| < \varepsilon \iff n > 4/\varepsilon$

So if $n > 4/\varepsilon$ and $n \ge 3$,

$$\frac{n^2+2n}{n^3-5} < \frac{4}{n} < \varepsilon$$

**Proof** that $\frac{n^2+2n}{n^3-5} \longrightarrow 0$ (finally! Whew!)

Given $\varepsilon > 0$, set $N = \max\{2, 3, 3, 4/\varepsilon\}$. Then $n > N \Rightarrow$

$$\left|s_n - s\right| = \left|\frac{n^2+2n}{n^3-5} - 0\right| \underset{n \geq 2}{=} \frac{n^2+2n}{n^3-5} \underset{n \geq 3}{<} \frac{4}{n} \underset{n > 4/\varepsilon.}{<} \varepsilon$$

This is all a special case of...

**Thm** Suppose $a_n \to 0$. If for $k > 0$, $m \in \mathbb{N}$ we have $|s_n - s| \leq k \cdot a_n$ for all $n > m$, then $s_n \to s$.

**Thm** Suppose $a_n \to 0$. If for $k > 0$, $m \in \mathbb{N}$ we have

$|s_n - s| < k \cdot a_n$ for all $n > m$, then $s_n \to s$.

**Notes** ① $$\Big|\underset{s_n}{\frac{n^2+2n}{n^3-5}} - \underset{s}{0}\Big| \le \underset{k}{4}\Big(\underset{a_n}{\frac{1}{n}}\Big) \quad (m=3)$$

② This is the Squeeze Thm!

If $0 \le f(n) \le g(n)$ and $\lim\limits_{n\to\infty} 0 \le \lim\limits_{n\to\infty} f(n) \le \lim\limits_{n\to\infty} g(n)$

$$0 \le \lim_{n\to\infty} f(n) \le 0$$

$$\Rightarrow \lim_{n\to\infty} f(n) = 0.$$

**Thm** Suppose $a_n \to 0$. If for $k>0$, $m \in \mathbb{N}$ we have $|s_n - s| < k \cdot a_n$ for all $n > m$, then $s_n \to s$.

**Pf.** Let $\varepsilon > 0$, need to show $|s_n - s| < \varepsilon$ for (eventual) values of $n$.

Because $a_n \to 0$, $\exists$ $N_1$ s.t. $|a_n - 0| = |a_n| < \varepsilon/k$.
Let $N = \max\{m, N_1\}$. Then $n > N \Rightarrow$

$$|s_n - s| \underset{n>m}{\leq} k|a_n| \underset{\text{b/c } n>N_1}{<} k(\varepsilon/k) = \varepsilon$$

Q: How to prove a sequence diverges ?

Recall the def$^{\underline{n}}$:

implied $\forall\, n>N$

$s_n \to s$ iff $\forall \varepsilon>0\ \exists N$ s.t. $n>N \Rightarrow |s_n - s| < \varepsilon$.

$s_n \not\to s$ iff $\exists \varepsilon>0 \ni \forall N\ \exists n \ni n>N$ and $|s_n - s| \geq \varepsilon$

i.e. $\exists\, \varepsilon>0$ for which you can't ever guarantee $s_n$ is ever within $\varepsilon$ of $s$.

Ex $\frac{1}{n} \not\to 9$ b/c we can't guarantee $\frac{1}{n}$ will be within $\varepsilon$ units of 9 for $\varepsilon = 1$. In fact $\left|\frac{1}{n} - 9\right| \geq \varepsilon = 1$ for all $n$ — not even a matter of finding $n$!

So to prove $s_n$ DIVERGES, you must show it cannot converge to any $s \in \mathbb{R}$ — that's hard!

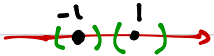
Show: $\forall s \ \exists \varepsilon > 0 \ni \forall N \ \exists n \ni n > N$ and $|s_n - s| \geq \varepsilon$

Must exhibit $\varepsilon > 0$ s.t. no matter how large $n$ gets, you cannot guarantee $|s_n - s| < \varepsilon$ for any $s$.

Ex Prove $s_n = (-1)^n$ diverges. $s_n = (-1, 1, -1, 1, -1, 1, \dots)$

Let $\varepsilon = 1/2$, suppose $s_n \to s$.
$\exists N$ s.t. $n > N$, $|s_n - s| < 1/2$.

Thus $-\frac{1}{2} < s_n - s < \frac{1}{2}$



No matter how large $n$ is, $s_n$ still osillates b/w $1, -1$.

**Ex** Prove $s_n = (-1)^n$ diverges.

n odd $\Rightarrow s_n = -1$, so $-\frac{1}{2} < -1 - s < \frac{1}{2} \Rightarrow -\frac{3}{2} < s < -\frac{1}{2}$

n even $\Rightarrow s_n = 1$, so $-\frac{1}{2} < 1 - s < \frac{1}{2} \Rightarrow \frac{1}{2} < s < \frac{3}{2}$

Thus $-\frac{3}{2} < s < -\frac{1}{2} < \frac{1}{2} < s < \frac{3}{2}$

i.e. $s < s$, a contradiction. Hence $s_n$ does not converge to any $s$.

($s_n$ diverges)

**Thm** If $s_n \to s$, $s_n \to t$, then $s=t$. (lim of a seq. is unique)

**Pf** We want to show $s=t \iff s-t=0$

$\iff |s-t|=0$

$\iff |s-t|<\varepsilon \; \forall \varepsilon>0$

Since $s_n \to s$, $s_n \to t$, given $\varepsilon>0$ we know

$\exists$ $M$ s.t. $n>M \Rightarrow |s_n - s| < \varepsilon/2$

$\exists$ $K$ s.t. $n>K \Rightarrow |s_n - t| < \varepsilon/2$

Then $N=\max\{M,K\} \Rightarrow$ both ineq's true.

For $n>N$ we therefore have...

$$\begin{aligned} |s-t| &= |(s_n-t)-(s_n-s)| && \text{(added } 0=s_n-s_n\text{)} \\ &\le |s_n-t| + |s_n-s| && (\triangle \text{ ineq}) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= 2\varepsilon/2 \\ &= \varepsilon \end{aligned}$$

as desired.

**Thm** A convergent sequence is bounded

Read in book with paper and pencil to draw and follow along.