§5.1 (Precise Def of) Limits of Functions

Today: $\lim_{x \to a} f(x)$

Why didn't we start with this, instead of $\lim_{n \to \infty} s_n$?

(1) Teaching: sequence defⁿ not as complicated

(2) Textbook defⁿ more complicated than needed.

You can use book as a reference, but anything you need to know for final will be covered here.

Wednesday: limit wrapup, prep for final exam.

From the beginning, "limits" and "f(x) getting really, really close to L" were derided.

Berkeley (18th Century)

infinitesmals are the "ghosts of vanishing quantities."

Russell (20th Century)

.. are "unnecessary, erroneous and self-contradictory"

Leibniz

Newton

Cauchy

Weierstrass

With sequences:

$\lim\limits_{n \to \infty} s_n = L$ if $\forall \varepsilon > 0$, $\exists\, N$ s.t. $n > N$ forces $|s_n - L| < \varepsilon$

i.e. $\underbrace{\text{eventually}}_{\exists N}$, $\underbrace{\text{every \# in}}_{n > N}$ seq. is ($\underbrace{\text{arbitrarily}}_{\forall \varepsilon > 0}$) $\underbrace{\text{close to } L}_{|s_n - L| < \varepsilon}$.

More generally, for $f: \mathbb{R} \to \mathbb{R}$,

$\lim_{x \to a} f(x) = L$ means

As $\underline{x}$ $\underbrace{\text{approaches}}_{?}$ a, $f(\underline{x})$ $\underbrace{\text{gets closer and closer to } L}_{|f(x) - L| < \varepsilon}$.

Precise Def: $\lim_{x \to a} f(x) = L$ means:

$$\forall \varepsilon > 0 \ \exists \ \delta > 0 \text{ s.t. } \underbrace{0 < |x-a| < \delta}_{\substack{x \text{ close} \\ \text{to } a, \text{ but} \\ \text{not equal} \\ \text{to } a.}} \Rightarrow \underbrace{|f(x) - L| < \varepsilon}_{\substack{f(x) \text{ close to } L \\ (\text{arbitrarily close} \\ \text{b/c of } \forall \varepsilon > 0)}}.$$

$\lim_{x \to a} f(x) = L$ if: $\forall\, \varepsilon > 0\ \exists\, \delta$ s.t. $0 < |x-a| < \delta$ forces $|f(x) - L| < \varepsilon$.

! As with sequnces, ORDER IS IMPORTANT.

ε is chosen first. Then you have to find δ to make def^n work.

As with limits of sequences, we'll use a "Think" step and a "Proof" step.

Ex Prove $f(x)=3x-1$ is continuous at $x=2$.

Need to show: $\lim_{x\to 2} f(x) = f(2)$, i.e. $\lim_{x\to 2} 3x-1 = 5$.

Think: Want to find δ s.t $|x-2|<\delta \Rightarrow |3x-1-5|<\varepsilon$.

Method 1

$$|3x-6|<\varepsilon$$
$$-\varepsilon < 3x-6 < \varepsilon$$
$$-\varepsilon < 3(x-2) < \varepsilon$$
$$-\varepsilon/3 < x-2 < \varepsilon/3$$
$$|x-2| < \varepsilon/3$$

Method 2

$$|3x-6|<\varepsilon$$
$$|3(x-2)|<\varepsilon$$
$$3\cdot|x-2|<\varepsilon$$
$$|x-2|<\varepsilon/3$$

Ex Prove $f(x) = 3x - 1$ is continuous at $x = 2$.

Need to show: $\lim_{x \to 2} f(x) = f(2)$, i.e. $\lim_{x \to 2} 3x - 1 = 5$.

Proof:

Let $\varepsilon > 0$ and choose $\delta = \varepsilon/3$. Then $0 < |x-2| < \delta \implies$

$$|(3x-1) - 5| = |3x - 6| = 3|x-2| < 3(\varepsilon/3) = \varepsilon$$

$\forall \varepsilon > 0 \; \exists \delta$ s.t. $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Ex Prove $\lim_{x \to 0} x^4 = 0$

T: Want to find δ s.t. $0<|x-0|<\delta$ forces $|x^4-0|<\varepsilon$.

$$|x^4| = |x|^4 = |x-0|^4 < \varepsilon \iff |x-0| < \varepsilon^{1/4}$$

P: Let $\varepsilon>0$. Choose $\delta = \sqrt[4]{\varepsilon} = \varepsilon^{1/4}$. Then $|x-0|<\delta$,

$$|x^4-0| = |x^4| = |x|^4 = |x-0|^4 < \left(\varepsilon^{1/4}\right)^4 = \varepsilon$$

$$\forall \varepsilon>0 \ \exists \delta \text{ s.t. } 0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$$

Ex Prove $\lim_{x \to 4} x^2-2x-3 = 5$

Think: want $0<|x-4|<\delta$ to force $|(x^2-2x-3)-5| < \varepsilon$.

We want $|x^2-2x-8| = |(x+2)(x-4)| = |x+2| \cdot |x-4| < \varepsilon$.

⚠ We can make $|x-4|$ small, but not $|x+2|$.... or can we?!
If $|x-4|$ really small, then $x \approx 4$ and $x+2 \approx 6$.

If I eventually make sure $\delta<1$, then

$$|x-4| < \delta = 1$$
$$-1 < x-4 < 1$$
$$3 < x < 5$$
$$5 < x+2 < 7$$
$$|x+2| < 7$$

So if $|x-4|<1$, $|x^2-2x-8| = |x+2|\cdot|x-4| < 7\cdot|x-4| < \varepsilon$

suggests $|x-4| < \varepsilon/7 = \delta$.

Proof: Let $\varepsilon>0$, and choose $\delta = \min\{1, \varepsilon/7\}$. Thus if $0<|x-4|<\delta$, then not only is $|x-4|<\varepsilon/7$, but also $|x-4|<1$

$$-1<x-4<1$$

$$\vdots$$

$$|x+2|<7.$$

Furthermore,

$$|(x^2-2x-3)-5| = |x^2-2x-8| = |x+2|\cdot|x-4| < 7(\varepsilon/7) = \varepsilon.$$

<u>Ex</u> Prove $\lim_{x\to -2} x^2 - 1 = 3.$

$$|(x^2-1)-3| = |x^2-4| = |x-2|\cdot|x+2|.$$

We can make $|x+2|$ small; if $|x+2|<1$, then

$$-1 < x+2 < 1$$
$$-5 < x-2 < -3$$
$$|x-2| < 5$$

Then $|x-2|\cdot|x+2| < 5\,|x+2| < \varepsilon$ suggest $\delta = \varepsilon/5.$

Let $\varepsilon < 0$.

<u>Proof</u> Let $\varepsilon > 0$. Choose $\delta = \min\{1, \varepsilon/5\}$. Then $0 < |x+2| < \delta$ means $|x+2| < \varepsilon/5$ and $|x+2| < 1$; the latter forces

$$-1 < x+2 < 1$$
$$-5 < x-2 < -3$$
$$|x-2| < 5$$

Thus:

$$\left|(x^2-1)-3\right| = \left|x^2-4\right| = |x-2| \cdot |x+2| < 5\left(\varepsilon/5\right) = \varepsilon.$$

Final Exam

☆ $\exists$ three past finals (with solutions) that will be posted online

Review guide posted today/tomorrow.