Thm If $\sum |a_n|$ converges, so does $\sum a_n$.

Notes (1) If $\sum |a_n|$ converges we say $\sum a_n$ converges absolutely

(2) If $\sum a_n$ converges but $\sum |a_n|$ diverges, then the series converges conditionally

Ex $\sum \frac{(-1)^{n+1}}{n}$ converges, $\sum \frac{1}{n}$ diverges.

So the alternating harmonic series converges conditionally.

Pf that $\sum |a_n|$ converges $\Rightarrow \sum a_n$ converges

Pf $\left.\begin{array}{l} s_n = a_1 + a_2 + \cdots + a_n \\ t_n = |a_1| + \cdots + |a_n| \end{array}\right\}$ given $\sum |a_n|$ conv's (t_n conv's) want to show s_n converges.

$a_n \le |a_n| \Rightarrow s_n \le t_n$. Comp. Test doesn't apply. !!
(a_n, s_n may not be nonnegative)

$\sum |a_n|$ converges $\Leftrightarrow t_n$ converges $\Leftrightarrow t_n$ Cauchy.
So we can make $t_n - t_m$ small.

Given $\varepsilon > 0$, $\exists N$ s.t. $n > m > N \Rightarrow$

$$|t_n - t_m| = \Big| \, |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| \, \Big| < \varepsilon.$$

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| \le \underbrace{|a_{m+1}| + |a_{m+2}| + \cdots + |a_n|}_{= |t_n - t_m| \, !} < \varepsilon$$

$\Rightarrow s_n$ Cauchy $\Rightarrow s_n$ conv's $\Rightarrow \sum a_n$ conv's.

Other Tests

Thm (Alternating Series Test) If $a_n \to 0$ and is decreasing then $\sum (-1)^{n+1} a_n$ converges.

Ex $\frac{1}{n} \to 0$, decr'g $\Rightarrow \sum (-1)^{n+1} \frac{1}{n}$ conv's.

Thm (Ratio Test)

(a) if $\lim \left| \frac{a_{n+1}}{a_n} \right| < 1$ then $\sum a_n$ converges abs'y.

(b) if $\lim \left| \frac{a_{n+1}}{a_n} \right| > 1$ then $\sum a_n$ diverges

(c) otherwise no info

Thm (Root Test)

(a) $\lim |a_n|^{1/n} < 1$ then $\sum a_n$ conv's absolutely

(b) $\lim |a_n|^{1/n} > 1$ then $\sum a_n$ diverges

(c) otherwise no info

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim |a_n|^{1/n}$$

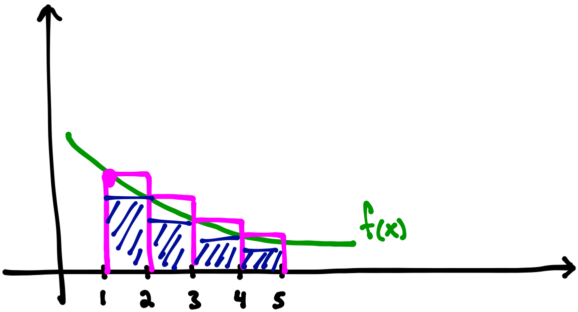
Ratio/Root Tests in §8.2 use lim inf and lim sup from §4.4 — which we don't cover. Use lecture versions for this class

<u>Thm</u> (Integral Test) Let $a_n = f(n)$, where $f:[0,\infty) \to \mathbb{R}$ is positive, continuous and decreasing. Then

$$\sum a_n = \sum f(n) \text{ converges} \iff \int_1^\infty f(x)\,dx \text{ converges}$$

<u>Recall</u>: $\displaystyle \int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b f(x)\,dx$

Sketch of Proof



Using left endpts, $a_1+a_2+a_3 \cdots + a_n \geq \int_1^n f(x)\,dx$
(b/c rectangles have more area than under curve)

if $\int_1^\infty f(x)\,dx = +\infty \Rightarrow \sum a_n = +\infty$ too.

Using right endpts, $a_1+\cdots+a_n \leq \int_1^n f(x)\,dx$ so if $\int_1^\infty f(x)\,dx$ converges, so does series.

$\lim \left|\frac{a_{n+1}}{a_n}\right|$, $\lim |a_n|^{1/n}$, $\lim \left(\int_1^n f(x)\,dx\right)$

$\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^n$ Ratio $\left|\frac{(3/(n+1))^{n+1}}{(3/n)^n}\right| = \frac{n^n}{(n+1)^{n+1}} \cdot \frac{3^{n+1}}{3^n} \to ?$ Root $\left|\left(\frac{3}{n}\right)^n\right|^{1/n} = \frac{3}{n} \to 0 \Rightarrow \sum \left(\frac{3}{n}\right)^n$ converges absolutely

$\sum_{n=1}^{\infty} \frac{3^n}{n!}$ Ratio $\frac{3^{\cancel{n+1}}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{3^n}} = \frac{3}{n+1} \to 0 < 1$ so conv. absolutely.

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ (Integral)

☆ $\sum_{n=1}^{\infty} \frac{n-1}{3n+2}$ $\frac{n-1}{3n+2} \to \frac{1}{3} \neq 0 \Rightarrow$ diverges.

$\sum_{n=1}^{\infty} \underline{(-1)^n} \frac{n^2+2}{3n^3}$ (use Alternating Series Test)

Ex More generally, $\sum \frac{1}{n^p}$ is called a **p-series**. Int test:

$$\int_1^n \frac{1}{x^p}\,dx = \int_1^n x^{-p}\,dx$$

(assume $p \neq 1$!)

$$= \frac{x^{-p+1}}{1-p}\Big|_1^n = \frac{n^{1-p}}{1-p} - \frac{1}{1-p}$$

As $n \to \infty$, convergence depends on n^{1-p} term.

converges if $1-p<0$ or $1<p$ (Ex $p=2$: $\frac{1}{n} \to 0$)

diverges if $1-p>0$ or $p<1$ (Ex $p=\frac{1}{2}$: $\sqrt{n} \to \infty$)

$p=1$: $\sum \frac{1}{n}$ harmonic series $\Rightarrow$ diverges.

Given $\sum a_n$, which test to use?

- If $a_n \not\to 0$, then $\sum a_n$ diverges.
- Known series: geometric series, p-series - use formulas
- [Check if it's telescoping]
- If a_n has $(-1)^n$ or $(-1)^{n+1}$ (etc), Alt. Series Test.
- If it's "similar" to known series (geo, p, harmonic): try comparison test:

Ex $\sum \frac{1}{3+2^n} \leq \sum \frac{1}{2^n}$

- Factorials, terms like 2^n, etc: Ratio Test.

- If n appears as exponent (and <u>especially</u> if it's also in base, e.g. n^n; <u>root</u> test