Math 3283W	**Name (Print):** Key & Grading Guide
Spring 2011	**Student ID:**
Final Exam	**Section Number:**
May 13, 2011	**Teaching Assistant:**
Time Limit: 120 minutes	**Signature:**

This exam contains 10 numbered problems. Check to see if any pages are missing. Point values are in parentheses. No books, notes, or electronic devices are allowed.

1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
6	20 pts	
7	20 pts	
8	20 pts	
9	20 pts	
10	20 pts	
TOTAL	200 pts	

1. (20 points) (4 points each) For each of the following five statements, determine whether the statement is true or false. Circle your answer. **No justification necessary.**

ALL OR NOTHING

(a) A conditional statement is logically equivalent to its contrapositive.

TRUE FALSE

(b) The set of rational numbers, together with the operations of addition and multiplication, is a complete ordered field.

TRUE FALSE not complete

(c) There exists a rearrangement of the terms of the alternating harmonic series that converges to zero.

TRUE FALSE

(d) For every sequence (a_n) of nonzero numbers, we have $\limsup |a_n|^{1/n} \leq \limsup |\frac{a_{n+1}}{a_n}|$.

TRUE FALSE

(e) Every member of the set $S = \{\frac{1}{n} : n \in \mathbf{N}\}$ is an isolated point of S.

TRUE FALSE

2. (20 points) (5 points each) Statements.

a. State the Principle of Mathematical Induction. (In your answer, let $P(n)$ denote a statement about the natural number n, and write your statement in the form "If ... and ..., then ....")

If $P(1)$ is true, — 1

and $P(k) \Rightarrow P(k+1)$ for all k, — 2

then $P(n)$ is true for all n. — 2

b. State the Intermediate Value Theorem.

2 1

If $f : [a, b] \to \mathbb{R}$ is continuous,

and if k is any number between $f(a)$ & $f(b)$,

then $\underbrace{\text{there exists } c}_{1}$ such that $\underbrace{a < c < b \text{ \& } f(c) = k}_{2}$.

c. State the Heine-Borel Theorem.

1

$S \subseteq \mathbb{R}$ is compact $\underbrace{\text{if and only if}}_{2}$ S is closed and bounded. 1

1

$\Rightarrow$ & $\Leftarrow$

d. State the Completeness Axiom.

1 1

If $S \subseteq \mathbb{R}$ is nonempty and bounded above,

then S $\underbrace{\text{has a supremum}}_{3}$.

3. (20 points) (5 points each) Definitions. Complete each sentence.

a. A sequence (a_n) *converges* to a if ...

$$\underbrace{\forall \varepsilon > 0}_{1}, \ \underbrace{\exists N}_{1} \text{ s.t. } \overbrace{\text{if } \overbrace{n \geq N}^{1} \text{ then } \underbrace{|a_n - a| < \varepsilon}_{1}}^{1}.$$

b. A series $\sum_{n=1}^{\infty} a_n$ *converges* to s if ...

$\underbrace{\text{the sequence } s_n = a_1 + \cdots + a_n \text{ of partial sums converges to } s}_{5}$, as above.

c. A number x is a *boundary point* of a set $S \subseteq \mathbf{R}$ if ...

1 — $\forall \varepsilon > 0$,

2 — $N(x, \varepsilon) \cap S \neq \emptyset$ and

2 — $N(x, \varepsilon) \cap S^c \neq \emptyset$.

d. Suppose that $S \subseteq \mathbf{R}$ is nonempty and bounded below. The real number m is the *infimum* of the set S if ...

- $m \leq s$, $\forall s \in S$. — 2

and
- if $m' \leq s$, $\forall s \in S$, then $m' \leq m$. — 3

(or contrapositive)

4. (20 points) (5 points each) Calculations. No justification necessary.

a. Find the limit of the convergent, recursively-defined sequence given by $a_1 = 5$ and $a_{n+1} = \sqrt{8a_n - 3}$.

the limit a satisfies $a = \sqrt{8a-3}$. — 2

1 → for identifying this one. $\Rightarrow a = 4 + \sqrt{13}$. 2 for solutions

($4 - \sqrt{13}$ is an extraneous root, because the sequence increases from 5)

b. Find the set S of subsequential limits and find $\limsup a_n$ and $\liminf a_n$ for the sequence

$$(a_n) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots\right).$$

3 → $S = \{0\}$.

$\underbrace{\liminf a_n}_{1} = \underbrace{\limsup a_n}_{1} = 0$. based on S.

c. Find the sum of the convergent series

$$\sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n.$$

(Note the index of the first term of the series. Write your answer in the form $\frac{m}{n}$, where m and n are natural numbers.)

$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$ ← 3

$\sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n = \frac{3}{2} - 1 - \frac{1}{3} - \frac{1}{9} = \frac{1}{18}$ ← 2

d. Write the boundary and interior of the set of rational numbers, as a subset of the real numbers.

$\partial \mathbb{Q} = \mathbb{R}$ — 3

$\text{int}\, \mathbb{Q} = \emptyset$ — 2

5. (20 points) (5 points each) Examples. No justification necessary. ALL OR NOTHING

a. Give an example of a series $\sum_{n=1}^{\infty} a_n$ that is convergent and has terms a_n that are constant.

$a_n \to 0$ is a necessary condition for convergence of $\sum a_n$.

The only sequence that is constant & converges to 0 is $\underline{a_n = 0}$. $\sum_{n=1}^{\infty} 0$

b. Give an example of a sequence (a_n) whose terms are all distinct and positive, and $\limsup a_n = +\infty$ and $\liminf a_n = 0$.

$(a_n) = (2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots)$

c. Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ and a subset $S \subseteq \mathbf{R}$ with the property that $f^{-1}(f(S)) \neq S$.

$f(x) = x^2$

$S = \{1\}$.

$f^{-1}(f(S)) = \{-1, 1\} \neq S$.

d. Give an example of an infinite collection $C_1, C_2, \ldots$ of closed subsets of $\mathbb{R}$ with the property that

$$\bigcup_{n=1}^{\infty} C_n$$

is not closed.

$C_n = [\frac{1}{n}, 3 - \frac{1}{n}]$ closed

$\bigcup_{n=1}^{\infty} C_n = (0, 3)$ not closed.

6. (20 points) Let A and B be sets, and let $f : A \to B$ be a function. Suppose that A_1 and A_2 are subsets of A. Prove that

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

Pf Let $y \in f(A_1 \cap A_2)$. 4

There exists $x \in A_1 \cap A_2$ with $f(x) = y$. 4

$x \in A_1 \Rightarrow y \in f(A_1)$. 4

$x \in A_2 \Rightarrow y \in f(A_2)$. 4

$\Rightarrow y \in f(A_1) \cap f(A_2)$. 4

7. (20 points) a. (15 points) Prove that for all n, we have

$$(2)(6)(10)(14)\cdots(4n-2) = \frac{(2n)!}{n!}.$$

When $n=1$, we have $2 = \frac{2!}{1!}$. — 3

Suppose that $(2)(6)\cdots(4k-2) = \frac{(2k)!}{k!}$, and show the statement is true for $n=k+1$. 3

$$(2)(6)\cdots(4(k+1)-2) = (2)(6)\cdots(4k-2)(4k+2)$$

$$= \frac{(2k)!}{k!}(4k+2) \quad \text{by hypothesis.} \quad 3$$

$$= \frac{(2k)!}{k!}\cdot 2\cdot(2k+1) = \frac{(2k+1)!}{k!}\cdot 2$$

$$= \frac{(2k+1)!}{k!}\cdot\frac{2(k+1)}{k+1} = \frac{(2k+2)!}{(k+1)!} \quad 3 \text{ algebra}$$

$$= \frac{(2(k+1))!}{(k+1)!}. \quad 2 \text{ form}$$

1 Thus, the statement is true for all n, by the principle of mathematical induction.

b. (5 points) Let us say that the *infinite product*

$$\prod_{i=1}^{\infty} a_i$$

has value P if the sequence (s_n) of partial products

$$s_n = \prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \cdots \cdot a_n$$

converges to P. Find the value of the infinite product

$$\prod_{n=1}^{\infty} \frac{1}{4n-2}.$$

By (a), we have $S_n = \frac{n!}{(2n)!}$. 1

2 reason

$\frac{S_{n+1}}{S_n} = \frac{n+1}{(2n+1)(2n+2)} \to 0$. By ratio test, $S_n \to 0$.

Thus $\prod \frac{1}{4n-2} = 0$. — 2

8. (20 points) a. (5 points) Complete the following definition: $f : \mathbf{R} \to \mathbf{R}$ is *continuous* at $x = a$ if ... (Note: this definition contains nested quantifiers. Also note that $\mathbf{R}$ is the domain of f.)

$\underbrace{\forall \varepsilon > 0}_{1}$, $\underbrace{\exists \delta > 0}_{1}$ such that

if $\underbrace{|x-a| < \delta}_{1}$ then $\underbrace{|f(x) - f(a)| < \varepsilon}_{1}$.

1

b. (5 points) Write the negation of your definition in (a).

$\underbrace{\exists \varepsilon > 0}_{1}$ such that $\underbrace{\forall \delta > 0}_{1}$, $\overbrace{\exists x \text{ such that}}^{1}$

$\underbrace{|x-a| < \delta}_{1}$ AND $\underbrace{|f(x)-f(a)| \geq \varepsilon}_{1}$.

c. (10 points) Show, directly from the definition of continuity, that the function $f(x) = x^2 - 3x$ is continuous at $x = 2$.

Note that on $(1,3)$, we have $|x-1| < 2$.

$\underbrace{\text{Let } \varepsilon > 0 \text{ be given.}}_{2}$ $\underbrace{\text{Choose } \delta = \min\{1, \tfrac{\varepsilon}{2}\}}_{2}$.

Then if $|x-2| < \delta$,

$$
\begin{aligned}
|f(x) - f(2)| &= |x^2 - 3x - (-2)| \\
&= |(x-2)(x-1)| \\
&= |x-2| \cdot |x-1| \\
&< 2|x-2|, \quad \text{since } \delta \leq 1. \\
&< 2\delta \leq 2 \cdot \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

2 algebra

2

2

9. (20 points) Find the set of all real numbers x for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x+5}{3}\right)^n.$$

Show your work.

If $b_n = \frac{1}{n}\left(\frac{x+5}{3}\right)^n$,

then $\left|\frac{b_{n+1}}{b_n}\right| = \frac{n}{n+1} \cdot \frac{1}{3}|x+5|$

$\rightarrow \frac{|x+5|}{3} < 1$?

5 Apply ratio/root test

find correct interval, modulo endpoints 5

if $-8 < x < -2$, convergence by ratio test

5 When $x=-2$, the series is a harmonic series that diverges.

5 When $x=-8$, it is an alternating harmonic series that converges.

The power series converges when $x \in [-8, -2)$.

Alternately:

$|b_n|^{1/n} = \frac{|x+5|}{3n^{1/n}} \rightarrow \frac{|x+5|}{3}$ & apply root test.

10. (20 points) (10 points each) Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Justify your answers. In the case of convergence, do not find the sum.

a.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^4+2}}$$

5

check inequality

$$|a_n| = \frac{n}{\sqrt{n^4+2}} > \frac{n}{\sqrt{n^4+2n^4}} = \frac{1}{n\cdot\sqrt{3}},$$

so $\sum |a_n|$ diverges by the comparison test.

1 mention

5

However, since $\frac{n}{\sqrt{n^4+2}} \to 0$ and is a decreasing

1 mention

function of n (no justification needed),

$\sum a_n$ converges by the alternating series test.

1 mention

Thus $\sum a_n$ converges conditionally.

b.

$$\frac{1}{2} - \frac{1}{2\cdot 3} + \frac{1}{2^2\cdot 3} - \frac{1}{2^2\cdot 3^2} + \frac{1}{2^3\cdot 3^2} - \cdots$$

5

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \frac{1}{3}, & \text{if } n \text{ is odd} \\ \frac{1}{2}, & \text{if } n \text{ is even.} \end{cases}$$

5

$\limsup \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1$ and hence

$\sum a_n$ converges absolutely by the ratio test.