The following is a partial list of solutions to homework problems, to help you double check answers as you prepare for the test. In some cases I might just give an answer; in others I might just give a few words of explanation. Sometimes you'll get both!

Please let me know if you spot a typo here!

1.11: There are 3^5 total functions from $\{1,2,3\}$ to $\{1,2,3,4,5\}$, because to define a function you need to determine $f(1)$, $f(2)$, and $f(3)$, and there are five choices of outputs for each function value. If f is injective, then each output has to be differerent, so there are $5 \cdot 4 \cdot 3 = 80$ possibilities. Note that there are *no* surjective functions, because with only three inputs, f can't possibly hit all five outputs.

1.31: $\sum_{i=4}^{i=10} \binom{10}{i} (\frac{3}{10})^i (\frac{7}{10})^{10-i} = 0.350\ldots$. This is also $1 - \sum_{i=0}^{i=3} \binom{10}{i} (\frac{3}{10})^i (\frac{7}{10})^{10-i}$, where you subtract the probability of 0, 1, 2, or 3 red balls from 1.

1.42: Following my hint, you can construct the following system of equations

$$\begin{aligned} E &= E_T \\ E_T &= p(1+E_H) + q(1+E_T) = 1 + pE_H + qE_T \\ E_H &= p(1) + q(1+E_T) = 1 + qE_T \end{aligned}$$

Solving this system leads to $E = \frac{1+p}{p^2}$, and with $p = \frac{1}{2}$, $E = 6$.

1.43: With $p = .11$ in the formula from 1.42, $E = 91.7355\ldots$.

2.02: 2.32393.

2.05: The hardest part of this problem is figuring out the probabilities that the sum is $3, 4, \ldots, 18$. That's harder than any of the counting problems on the test, so I'll just tell you that the probabilities are

$$\begin{aligned} &1/216, 1/72, 1/36, 5/108, 5/72, 7/72, 25/216, 1/8, \\ &1/8, 25/216, 7/72, 5/72, 5/108, 1/36, 1/72, 1/216 \end{aligned}$$

Given that distribution, the entropy is about 3.6.

2.A: The main part here is that $P(X = x) = 1$ for some x, and the probability that X equals anything else is zero.

3.02: Follow the hint for 3.01 in the back of the book.

3.05: Including trees is not so easy, but you can check your process with me. I got the following code. If you have something different, *it might still be correct*, because you have to make ambiguous choices along the way. There are 6 things to be encoded; I'll call them

A, B, C, D, E, and F.

$$f(A) = 00$$
$$f(B) = 01$$
$$f(C) = 10$$
$$f(D) = 110$$
$$f(E) = 1110$$
$$f(F) = 1111$$

As I recall, the average word length is 2.375, and the entropy is about 2.32. The entropy has to be less than or equal to word length by the Noiseless Coding Theorem, so that turns out as expected.

3.06: Similar to 3.05

3.A: There doesn't, because of the Kraft-McMillan Inequalities. If $|\Sigma| = 3$ then it's possible.

3.B: The first and third are instantaneous. The second is not, because any codeword is not only the prefix of some other codeword – it's the prefix of *every* following codeword on the list!

4.04: The probability of at least one error in the transmission of four digits is the same as

$$\begin{aligned} 1 - P(\text{no errors}) &= 1 - \left(\frac{3}{4}\right)^4 \\ &= 1 - \frac{81}{256} \\ &= 0.683594\ldots \end{aligned}$$

The parity bit will detect a 1- or 3-bit error, so the probability of an undetected error is the probability of a 2- or 4-bit error.

$$\begin{aligned} P(2\text{ errors}) + P(2\text{ errors}) &= \binom{4}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 + \binom{4}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^0 \\ &= 6 \cdot \frac{1}{16}\frac{9}{16} + 1 \cdot \frac{1}{256} \cdot 1 \qquad = \frac{55}{256} = 0.214844\ldots \end{aligned}$$

4.06: On an intuitive level, any channel with $p > 1/2$ is so unreliable we shouldn't use it. However, on a more abstract level, it's ok to use such a channel, but we don't learn anything new that we don't get from considering channels with $p < 1/2$. For example, if $p = .001$, I can be 99.7% sure that if I receive 000 it was sent as 000. If $p = .999$, on the other hand, I can be 99.7% sure that if I receive 000, it started off as 111 – every bit was most likely flipped! The key is that the behavior is *predictable*, so I can account for it. High values of p are just as predictable as low values, but the bits get flipped instead of being left alone.

4.11: The rate is $R = \frac{\log_2 16}{12} = \frac{1}{3}$

4.A: This channel is useless, because we're essentially told that the probability of receiving 0 is p (and the probabiliy of receiving 1 is $1 - p$), regardless of what's being sent!