Math 5335
Fall 2018
Exam 2
12/9/19
Time Limit: 90 Minutes

Name (Print): KEY

This exam contains 9 pages (including this cover page) and 13 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and **put your initials** on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you are applying a theorem, you should indicate this fact**, and explain why the theorem may be applied.
- **Do not trivialize a problem**. If you are asked to prove a theorem, you cannot just cite that theorem.
- **Organize your work** in a reasonable, tidy, and coherent way. Work that is disorganized and jumbled that lacks clear reasoning will receive little or no credit.
- **Unsupported answers will not receive full credit.** An answer must be supported by calculations, explanation, and/or algebraic work to receive full credit. Partial credit may be given to well-argued incorrect answers as well.
- If you need more space, use the back of the pages. **Clearly indicate when you have done this.**

Do not write in the table to the right.

Page	Points	Score
2	10	
3	12	
4	9	
5	9	
6	15	
7	22	
8	10	
9	13	
Total:	100	

You may use the following results on the exam without defining or proving them.

The distance between points (a, b) and (a, d) in the Poincaré Half Plane is $|\ln(d/b)|$.

The distance between points P_1 and P_2 on the line $(x-\omega)^2 + y^2 = \rho^2$, with angles $t_1 = |\angle(\omega+\rho, 0)(\omega, 0)P_1|$ and $t_2 = |\angle(\omega+\rho, 0)(\omega, 0)P_2|$ is

$$\ln\left[(\csc t_2 - \cot t_2) \,/\, (\csc t_1 - \cot t_1)\right]$$

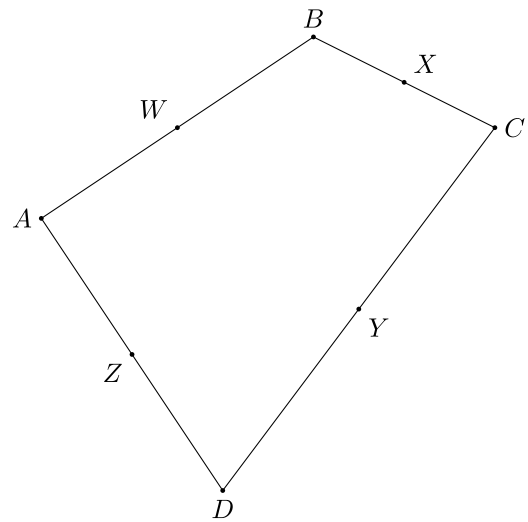
1. Let $ABCD$ be a convex, simple quadrilateral.

   (a) (4 points) Suppose $A + C = B + D$. Prove the opposite sides of $ABCD$ are parallel (which means $ABCD$ is a parallelogram).

   $A+C=B+D \Rightarrow B-A=C-D$, so two of the opposite sides represented by same (hence $\parallel$) vector. Thus $\overline{AB} \parallel \overline{CD}$

   Similarly: $A+C=B+D \Rightarrow D-A=C-B$, so $\overline{AD} \parallel \overline{BC}$

   (b) (6 points) Let W, X, Y and Z be the midpoints of the sides of $ABCD$, as shown in the generic diagram below. Prove that $WXYZ$ is a parallelogram.

   $W = \frac{A+B}{2}$

   $X = \frac{B+C}{2}$

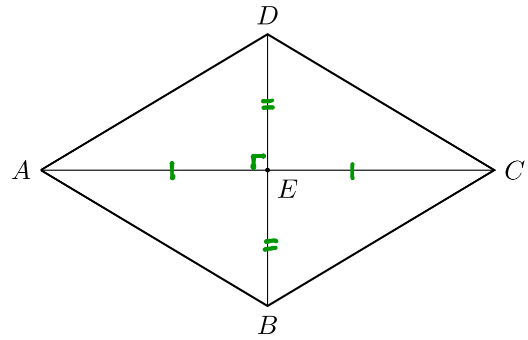
   $Y = \frac{C+D}{2}$

   $Z = \frac{A+D}{2}$

   $W+Y = \frac{A+B+C+D}{2} = X+Z$

   Thus $WXYZ$ is a $\parallel$'gram (by above)

2. (6 points) Let $ABCD$ be a parallelogram with perpendicular diagonals. Prove $ABCD$ is a rhombus (i.e. all four side of the quadrilateral are congruent).

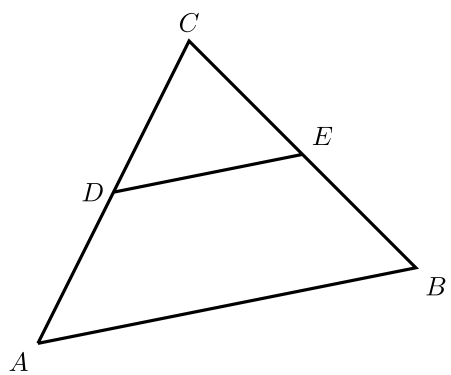


Diag's of a ||gram bisect each other.

Hence $\triangle AED \cong \triangle AEB \cong \triangle CEB \cong \triangle CED$ (by SAS)

$\Rightarrow \overline{AD} \cong \overline{AB} \cong \overline{CD} \cong \overline{CE}$

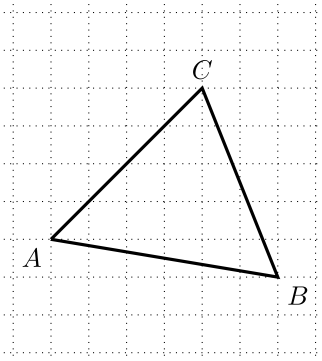
3. (6 points) Given $\triangle ABC$, let D and E be the midpoints of $\overline{AC}$ and $\overline{BC}$ as shown. Using any appropriate methods from the course, prove $\overline{DE} \parallel \overline{AB}$ and $|\overline{DE}| = \frac{1}{2}|\overline{AB}|$



$E - D = \frac{(B+C)}{2} - \frac{(A+C)}{2} = \frac{1}{2}(B-A).$

Thus $\overline{DE} \parallel \overline{AB}$ and $|\overline{DE}| = \frac{1}{2}|\overline{AB}|$

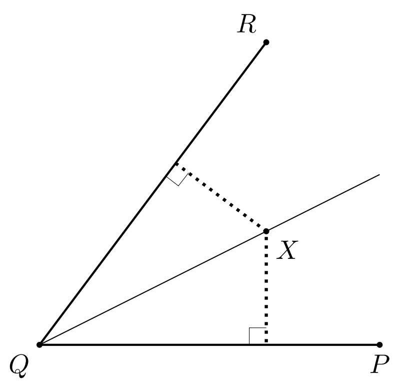
4. (3 points) Let $A = (0,3)$, $B = (6,2)$ and $C = (4,7)$. Find the centroid of $\triangle ABC$, in rectangular coordinates.

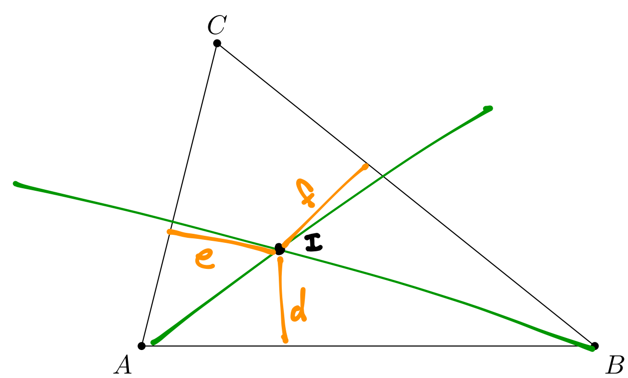


$$G = \frac{1}{3}(A+B+C) = \left(\frac{0+6+4}{3}, \frac{3+2+7}{3}\right) = \left(\frac{10}{3}, 4\right)$$

5. (6 points) Recall that the angle bisector of $\angle PQR$ is the set of all points which are equidistant to both sides of the angle; see the picture below on the left, where X is on the bisector of $\angle PQR$; the dotted lines from X are perpendicular to the sides, and are congruent.

Prove that the three angle bisectors of $\triangle ABC$ all intersect in a point I which is equidistant to all three sides. (You can use the picture below on the right for your conenience but your argument must apply to all triangles.)





Let I be intersection of angle bisectors of $\angle BAC$ and $\angle ABC$. Thus

- I equidistant to $\overline{AB}$ and $\overline{AC}$. (In pic, $d=e$)
- I —— " —— $\overline{AB}$ and $\overline{BC}$. ($d=f$)

By transitivity, $d=e=f$ and I equidistant to all 3 sides.

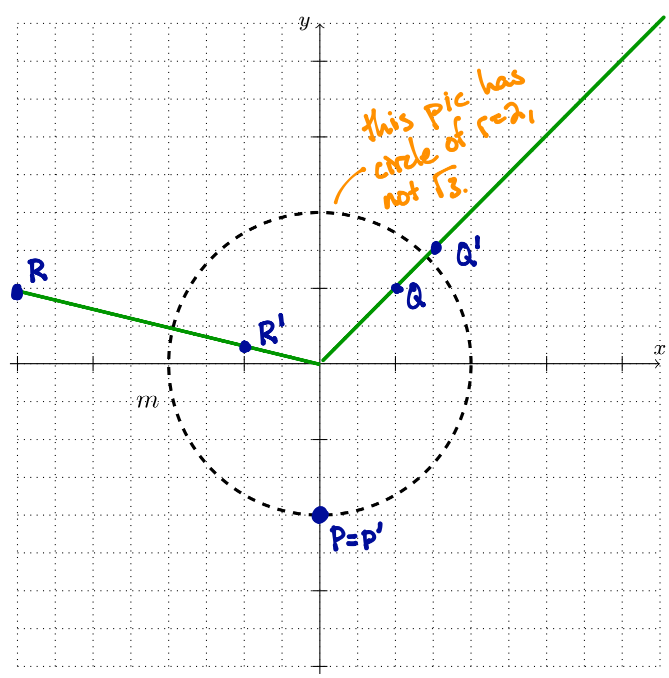
6. (3 points) Let A and B be distinct points in $\mathbb{R}^2$. Describe all points $C \in \mathbb{R}^2$ for which there exists a circle through A, B and C.

$C \notin \overleftrightarrow{AB}$.

(b/c then A, B, C not collinear, so can construct circumcircle of $\triangle ABC$.)

(Also, $C = A$ or $C = B$ allows for circle through A, B, but this case wasn't required.)

7. (6 points) Let $m = \{\|X\| = \cancel{2}\ \sqrt{3}\} = \{(x, y) : x^2 + y^2 = 3\}$. Find the coordinates of the reflection of each of the following points across m. Graph the original points P, Q and R, and their reflections P', Q', and R'. *Note that the gridlines are drawn every half unit in this picture.*



- $P = (0, -\sqrt{3})$ $= (0, -\sqrt{3})$. on mirror, so fixed.

- $Q = (1, 1)$

Want $|\overline{OQ}| \cdot |\overline{OQ'}| = \sqrt{2} \cdot |\overline{OQ'}| = \rho^2 = 3$

$|\overline{OQ'}| = \frac{3}{\sqrt{2}}$

$Q' = \underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}_{\text{unit DI for ray.}} \cdot \frac{3}{\sqrt{2}} = \left(\frac{3}{2}, \frac{3}{2}\right)$

- $R = (-3, 1)$

Want $|\overline{OR}| \cdot |\overline{OR}| = 3$

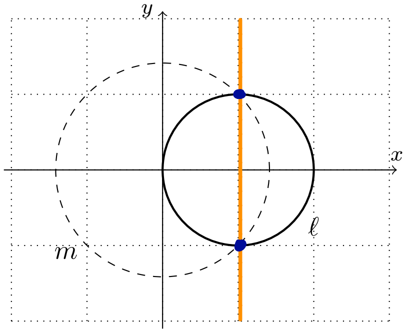
$\sqrt{10} \cdot |\overline{OR'}| = \frac{3}{\sqrt{10}}$

(or could use $\frac{\rho^2}{\|X\|^2} X$ formula)

$R' = \left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) \cdot \frac{3}{\sqrt{10}} = \left(-\frac{9}{10}, \frac{3}{10}\right)$

8. In each part below, sketch ℓ', the reflection (i.e. inversion) of ℓ across the mirror m. Also give the equation for ℓ'.

(a) (5 points) $m : x^2 + y^2 = 2$ and $\ell : (x-1)^2 + y^2 = 1$.

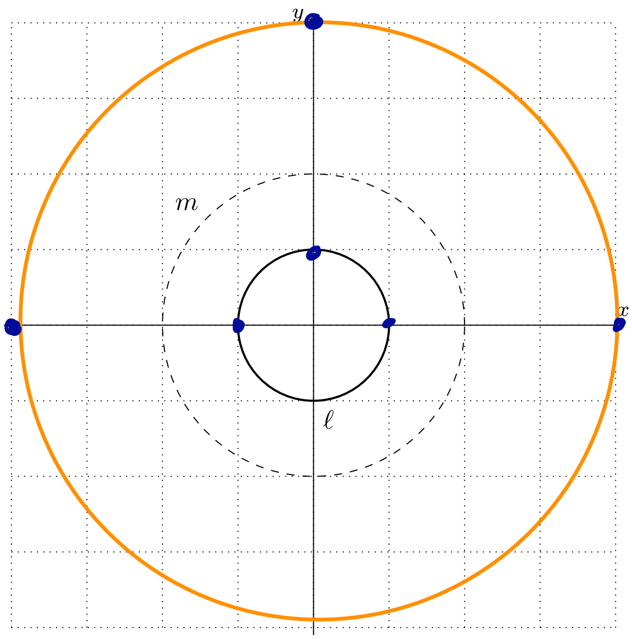


$\ell \cap m = \{(1, \pm 1)\}$ are fixed, hence on ℓ'.

$0 \in \ell \Rightarrow \infty \in \ell$, so ℓ is a line through $(1, \pm 1)$

$\ell' : x = 1$

(b) (5 points) $m : x^2 + y^2 = 4$ and $\ell : x^2 + y^2 = 1$.



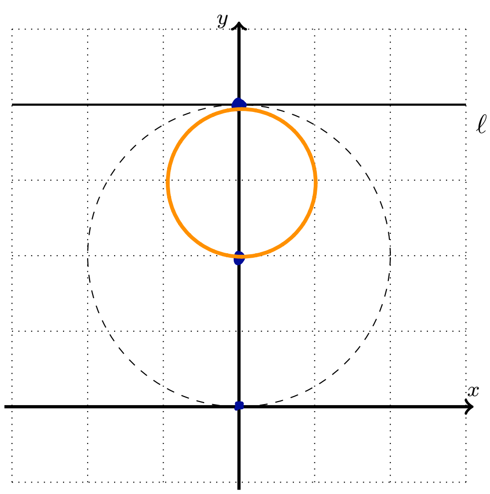
$0 \notin \ell$ so $\infty \notin \ell' \Rightarrow \ell'$ a circle.

$\infty \notin \ell \Rightarrow 0 \notin \ell \Rightarrow \ell'$ a circle not through 0

Can reflect 3 pts (e.g. $(\pm 1, 0)$, $(0,1)$)

to find that $\ell' : x^2 + y^2 = 16$.

(c) (5 points) $m : x^2 + (y-2)^2 = 4$ and $\ell : y = 4$.



center of m

$\infty \in \ell \Rightarrow (0,2) \in \ell'$

$0 \notin \ell \Rightarrow \infty \notin \ell'$ so ℓ' a circle through origin.

$(0,4) \in m$, hence fixed. Segment from $(0,2)$ to $(0,4)$

becomes the diameter

$\ell' : x^2 + (y-3)^2 = 1$

m

On this page, all points, lines, segments and distances are in the **Poincaré Half Plane**.

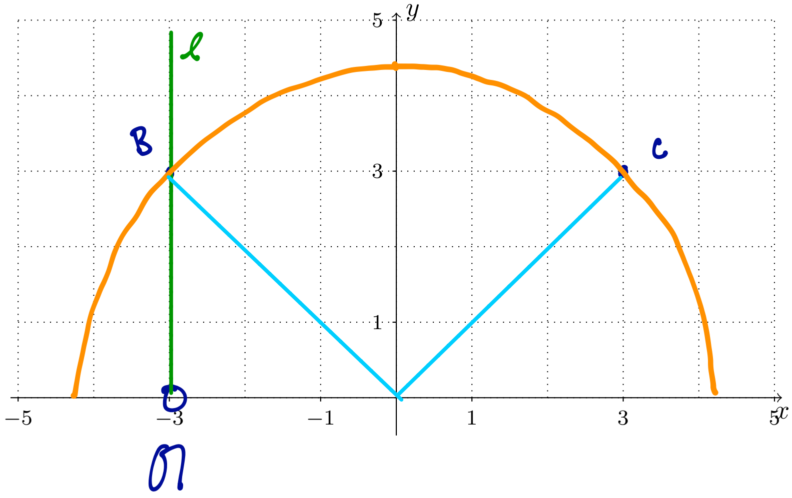
9. Let $\mathfrak{A} = (-3, 0)$, $B = (-3, 3)$, and $C = (3, 3)$.

(a) (10 points) Sketch the line ℓ through B directed by $\mathfrak{A}$, and the line $\overleftrightarrow{BC}$. Find the equations of both lines.

ℓ: $x = -3$

$\overleftrightarrow{BC}$: $|\overline{0C}| = \sqrt{18} = 3\sqrt{2}$ (≈ 4.25 ish)

$x^2 + y^2 = 18$



(b) (3 points) Find the length of segment $\overline{BC}$.

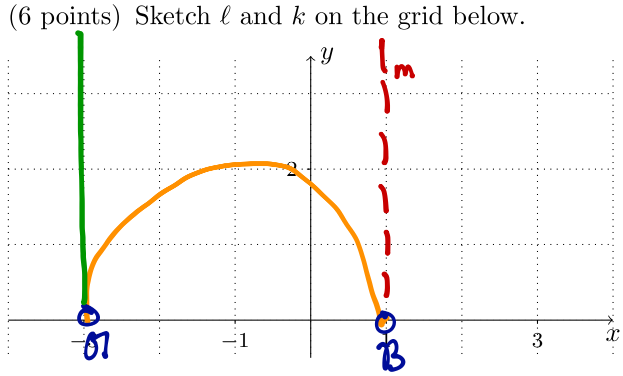
$\sin 3\pi/4 = \frac{1}{\sqrt{2}}$

From front: $d(B,C) = \ln \left| \dfrac{\csc 3\pi/4 - \cot 3\pi/4}{\csc \pi/4 - \cot \pi/4} \right|$

$= \ln \left| \dfrac{\sqrt{2}+1}{\sqrt{2}-1} \right|$ $\left(= \ln(3 + 2\sqrt{2})\right.$. Lots of equiv. answers.)

10. Let ℓ be the line directed by $\mathfrak{A} = (-3, 0)$, $\mathfrak{B} = (1, 0)$, and k be the line directed by $\mathfrak{A}$ and $\mathfrak{C} = (\infty, 0)$.

(a) (6 points) Sketch ℓ and k on the grid below.



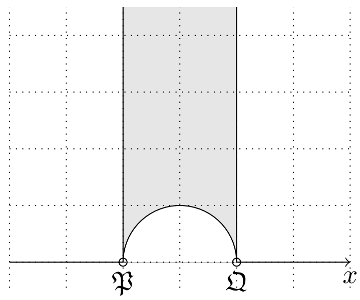
(b) (3 points) Give an equation for a line ~~k~~ m containing $(1, 2)$ which is *asymptotically* parallel to both ℓ and k, or explain why none exists.

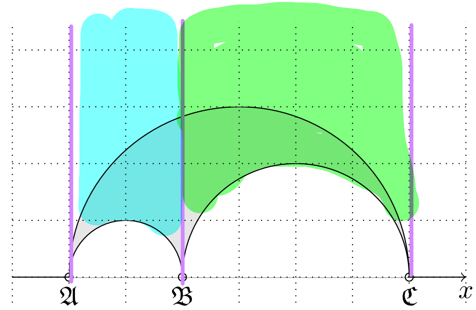
m : $x = 1$ works; shares a PDI w/ both.

(other answers possible, e.g. dir'd by $\mathfrak{A}$ and $(d, 0)$, $d > 1$.)

On this page, all points, lines, segments, triangles and areas are in the **Poincaré Half Plane**.

11. (a) (5 points) The area of any triply asymptotic triangle with Poincaré Direction Indicators $\mathfrak{P} = (p, 0)$, $\mathfrak{Q} = (q, 0)$ and $\mathfrak{R} = (\infty, 0)$, as shown on the left, is π. Use this fact to prove the area of the triply asymptotic triangle $\triangle\mathfrak{ABC}$ on the right is also π.





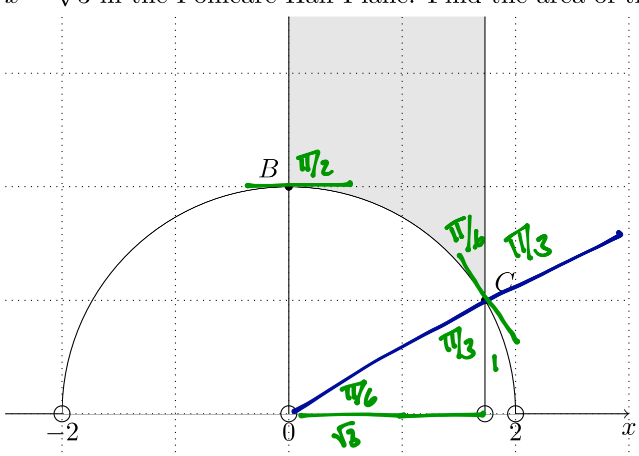
Let $\mathfrak{D} = (\infty, 0)$.

$$\|\triangle\mathfrak{ABC}\| = \|\triangle\mathfrak{ABD}\| + \|\triangle\mathfrak{BCD}\| - \|\triangle\mathfrak{ACD}\|$$

$$= \pi + \pi - \pi$$

$$= \pi$$

(b) (5 points) The picture below shows a singly asymptotic triangle $\triangle\mathfrak{A}BC$ formed by the lines $x^2 + y^2 = 4$, $x = 0$ and $x = \sqrt{3}$ in the Poincaré Half Plane. Find the area of the triangle using any valid method.



$$\|\triangle\mathfrak{A}BC\| = \pi - \beta - \gamma$$

$$= \pi - \frac{\pi}{2} - \frac{\pi}{6}$$

$$= \frac{\pi}{2} - \frac{\pi}{6}$$

$$= \pi/3$$

12. For each of the following statements, indicate whether it is **True** or **False** by circling the corresponding answer. Briefly justify your answer.

(a) (4 points) Lines $x = \lambda$ and $x = \mu$ in the Poincaré Half Plane are always ultra parallel.

True **False**

never ultra-$\parallel$. Always share $(\infty, 0)$.

(b) (4 points) $\mathcal{D}(x, y) = (4x, 4y)$ is a conformal affinity.

True **False**

$\mathcal{D}(x,y) = 4\,\underbrace{\mathcal{I}(x,y)}_{\text{identity (an isometry)}}$

13. (5 points) Recall the incidence axioms from Chapter 10:

I.1 For any two distinct points P and Q, there exists a unique line that is incident with both P and Q.

I.2 Every line is incident with at least two points.

I.3 There exist three points such that no line is incident with all three.

Define an *8 point geometry* as follows. The *points* are the vertices of a unit cube in $\mathbb{R}^3$, as shown below. A line is defined to be a set of four points which form the vertices of a square face of the cube. For example, $\{(0, 0, 0), (1, 0, 0), (1, 1, 0), (0, 1, 0)\}$ is a line because those points are the vertices of the square on the bottom of the cube.

Does this 8 point geometry satisfy the incidence axioms? Justify your answer.

No – no line incident w/ $(0,0,0)$ and $(1,1,1)$, for example, so fails I.1.